The Enumerative Significance of Elliptic Gromov-Witten Invariants in $\mathbb{P}^3$

Master Thesis

Department of Mathematics
Technische Universität Kaiserslautern

Author:
Felix Röhrle

Supervisor:
Prof. Dr. Andreas Gathmann

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Selbstständigkeitserklärung

Hiermit bestätige ich, Felix Röhrle, dass ich die vorliegende Masterarbeit zum Thema „The Enumerative Significance of Elliptic Gromov-Witten Invariants in $\mathbb{P}^3$" selbstständig und nur unter Verwendung der angegebenen Quellen verfasst habe. Zitate wurden als solche kenntlich gemacht.

Felix Röhrle
Kaiserslautern, Mai 2018

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Introduction

Enumerative geometry is characterized by the question for the number of geometric objects satisfying a number of conditions. A great source for such problems is given by the pattern "how many curves of a given type meet some given objects in some ambient space?". Here, one might choose to fix a combination of genus and degree of the embedding of the curve to determine its "type", and as "objects" one could choose points and lines. However, the number of objects is not free to choose – it should be determined in such a way that the answer to the question is a finite integer. For example one might ask [KV03]:

> *How many plane rational curves of degree d pass through $3d-1$ points in general position?*

This question has successfully been answered and the solution is quite illustrative for how enumerative problems might be approached. The first step is to set up a moduli space parametrizing the objects of interest. By means of suitable morphisms the point incidence conditions are pulled back from $\mathbb{P}^2$ to this moduli space. The resulting subvarieties are then intersected using the calculus of intersection theory to yield a zero-dimensional Chow cycle including multiplicities. A simple degree evaluation then gives rise to integers which have been found to indeed answer the question asked above.

The numbers obtained in the way just described can be defined in more general settings and are referred to as *Gromov-Witten invariants.* However, passing to positive genus introduces new phenomena causing a discrepancy between the invariants and the number sought by the enumerative question. Consider for example the problem:

> *How many elliptic curves of degree d are there which meet a number of a given lines and b given points in $\mathbb{P}^3$ in general position, where $4d = a + 2b$?*

Gromov-Witten invariants in the elliptic case can still be computed, but they are not answering the question. In fact, they are not even integers! In his 1996 paper [Get97, theorem 6.1] Getzler claims the answer to this question to be given by the formula

$$N_{ab}^{(1)} + \frac{1}{12}(2d-1)N_{ab}^{(0)},$$

where $N_{ab}^{(1)}$ and $N_{ab}^{(0)}$ are elliptic and rational Gromov-Witten invariants respectively. However, to the best of our knowledge no proof for the statement has been made public. Therefore we made it the goal of this thesis to prove Getzler's claim.

We now present an outline of our proceeding. In chapter 1 we will formally define the moduli space $\overline{M}_{g,n}(X, d)$ of so-called *stable maps* of genus g with n marks and degree d (definition 1.9) – a notion due to Kontsevich. In this notation X is the ambient space which we will take to be $\mathbb{P}^3$ in the end. This moduli space naturally comes with so-called *evaluation maps* $ev : \overline{M}_{g,n}(X, d) \to X$ (see definition 1.24) that enable us to translate incidence conditions in X into Chow cycles of the moduli space:

$$V \subseteq X \leftrightsquigarrow ev^*V \subseteq \overline{M}_{g,n}(X, d).$$

With this technique at hand we may introduce Gromov-Witten invariants in definition 1.25 by intersection these cycles and finally we restate Getzler's claim as main result of this thesis in theorem 1.28. However, the definition of concepts like *virtual dimension* and *virtual fundamental class* even though important will be largely omitted.

The discrepancy between Gromov-Witten invariants and enumerative numbers arises from a phenomenon referred to as *excessive dimension*. This is caused by the compactification of the moduli space $M_{g,n}(X, d)$ of smooth genus g curves that was defined in chapter 1. In chapter 2 we will explore the extent of the problem by determining all irreducible components of the moduli space of elliptic curves in $\mathbb{P}^3$ whose dimension exceeds the expected value (corollary 2.14). In order to do so we will prove some theorems constituting a dimension calculus for irreducible components (theorems 2.4 and 2.12).

In the third and final chapter we determine for each of the components found in chapter 2 the impact they have on the Gromov-Witten invariants. We start by identifying most of the excessive dimensional components as *enumeratively irrelevant* in theorem 3.7. This is done by constructing a morphism to a space of sufficiently small dimension which still keeps all of the enumerative information. Then we utilize the projection formula from intersection theory to move the computation of Gromov-Witten invariants from the original moduli space to the condensed space, where the result will be trivial for dimensional reasons.

Section 3.2 is dominated by the central computation of this thesis. We use the calculus of Chern classes to actually compute a virtual fundamental class of one of the excessive dimensional components. The result is expressed via tautological classes of the moduli spaces involved (theorem 3.21). In theorem 3.22 the coefficient claimed by Getzler shows up for the first time.

Finally in section 3.3 we use the compatibility of virtual fundamental classes with pull-back along forgetful morphisms to prove the main theorem.

Careful investigation of the central theorems in this thesis shows why we chose the special case of genus 1 curves in $\mathbb{P}^3$: In order for the Gromov-Witten invariants to have a chance of being enumerative we need $\dim M_{g,n}(X, \beta) = \operatorname{vdim} \overline{M}_{g,n}(, \beta)$, otherwise the virtual fundamental class restricted to $M_{g,n}(X, \beta)$ cannot be equal to the usual fundamental class. However our proof of theorem 2.12 works only for genus at most 1.

As for the choice of an ambient space X we restricted ourselves to $\mathbb{P}^r$ because we needed to use smoothness and the Euler sequence at various occasions. The choice of $r = 3$ is a bit more subtle however. Looking at the expression for the virtual dimension

$$\operatorname{vdim} \overline{M}_{g,n}(X,d) = -K_X d + (\dim X - 3)(1-g) + n$$

we see that it is independent of g only if $\dim X = 3$. This independence is needed to state theorem 3.22, i.e. in order to be able to express the push-forward of $[\overline{M}_{1,n}(X,d)]^{virt}$ along the map used in theorem 3.22 as a multiple of $[\overline{M}_{0,n}(X,d)]^{virt}$. This does of course not rule out the possibility of generalization, however if either of the constraints should be relaxed, different (and most likely more involved) tools will have to be employed.

Notation

Throughout this thesis, whenever we say “scheme” we mean “k-scheme”, where k is an algebraically closed, fixed base field of characteristic 0. Correspondingly all morphisms are morphisms of k-schemes. A variety is a separated scheme of finite type.

Curly capitals will always denote sheaves, fraktur font indicates categories. For example $\mathfrak{Mod}(R)$ is the category of modules over the ring R and $\mathfrak{Ab}$ is the category of Abelian groups.

$\underline{x}$	when the number of variables is clear from the context or irrelevant we will write this instead of $x_1, \ldots, x_N$
$()^\vee$	dual module, vector space, sheaf, etc.
$\mathbb{P}^r$	r-dimensional projective space, $\mathbb{P}^r := \operatorname{Proj} k[x_0, \ldots, x_r]$
$\mathcal{O}_X$	structure sheaf of a scheme or variety X
$\mathcal{O}_X(D)$	sheaf corresponding to the divisor $D \in \operatorname{Div}(X)$
$\mathcal{O}_{\mathbb{P}^r}(n)$	Serre twisting sheaf; $\mathcal{O}_{\mathbb{P}^r}(n) := \mathcal{O}_{\mathbb{P}^r}(n \cdot H)$, where $H \subseteq \mathbb{P}^r$ denotes a hyperplane
$H^i(X, \mathcal{F})$	i-th cohomology k-module of the scheme X with coordinates in the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$
$h^i(X, \mathcal{F})$	$:= \dim H^i(X, \mathcal{F})$
$A_i(X)$	group of i-dimensional Chow cycles in X
$A^i(X)$	group of Chow cycles of codimension i in X
$A^*(X)$	Chow ring non-singular, quasi-projective variety X, $:= \bigoplus_{i=0}^{\infty} A^i(X)$ with ring structure given by intersection product
$V \cdot W$	intersection product of $V, W \in A^*(X)$
$c_i(\mathcal{E})$	i-th Chern class of a vector bundle $\mathcal{E}$
$c(\mathcal{E})$	total Chern class, $c(\mathcal{E}) := \sum_{i=0}^{\infty} c_i(\mathcal{E}) \in A^*(X)$
$g(C)$	arithmetic genus of the curve C, $g(C) := h^1(C, \mathcal{O}_C)$
Ω_X	cotangent sheaf of X
T_X	tangent sheaf of X, $T_X := \Omega_X^\vee$
ω_X	canonical bundle of non-singular variety X, $\omega_X := \bigwedge^{\dim X} \Omega_X$
K_X	canonical divisor of X, $\mathcal{O}_X(K_X) = \omega_X$
$\Gamma(U, \mathcal{F})$	$:= \mathcal{F}(U)$, sections of the sheaf $\mathcal{F}$ on $U \subseteq X$ open
$\Gamma(\mathcal{F})$	$:= \mathcal{F}(X)$, global sections of $\mathcal{F}$
$\mathcal{F}_x$	$:= \varinjlim_{x \in U} \Gamma(U, \mathcal{F})$ the stalk of the sheaf $\mathcal{F}$ at the point $x \in X$

Chapter 1

The Moduli Space $\overline{M}_{g,n}(X, \beta)$

As very first step towards making any enumerative question accessible, one has to parametrize the objects of interest. In our case the objects of interest are curves of genus 1 in $\mathbb{P}^3$, and already here there are many possible models for such objects. We will work with a *parametrized* approach as opposed to an *embedded* approach – for a survey on the underlying ideas see [PT14]. More concretely this means that we will be working with stable maps and parametrize them by the moduli space $\overline{M}_{g,n}(X, \beta)$ which is due to Kontsevich. Defining all the notions involved and quoting the known existence results will be the first section of this chapter and follows [FP95].

The moduli spaces constructed in section 1 are by design compact. This means that they necessarily contain points which do not correspond to smooth but rather reducible curves. These points are said to lie in the boundary of $\overline{M}_{g,n}(X, \beta)$. In the second section we investigate the combinatoric structure of these curves.

After the moduli space has been established we move on to intersection theory on that space. In the third section we will introduce Gromov-Witten invariants which arise as intersection numbers. We will then state the main theorem of this thesis about the number of elliptic space curves meeting a certain number of lines and points.

Finally, in the last section we introduce some special Chow classes of our moduli space, the so-called *psi-* and *lambda-classes*. These will be used in Chapter 3 to compute virtual fundamental classes and ultimately to prove the main theorem.

1.1 Stable Curves and Maps

Definition 1.1. Let C be a projective, connected, reduced curve which has nodal singularities at worst, and let $C_1, \ldots, C_N$ denote the irreducible components of C. In order to distinguish between the C_i and other irreducible components we will refer to them as **twigs** of C, in analogy to [KV03]. Note that the C_i need not be smooth but may be nodal themselves.

Furthermore let $p_1, \ldots, p_n \in C$ pairwise different smooth points, called **marked points** or simply **marks**. The tuple $(C, p_1, \ldots, p_n)$ is said to be a **marked** or **n-pointed curve**. Whenever the marks are clear from context we will abuse notation and simply denote the marked curve by C again.

An **automorphism** of $(C, \underline{p})$ is an element $f \in \operatorname{Aut}(C)$ such that $f(p_i) = p_i$ for all marked points p_i.

A point $p \in C$ is called **special** if it is a mark or a singularity of C.

We say that $(C, \underline{p})$ is **stable** if all twigs with arithmetic genus 0 contain at least three special points and every twig of arithmetic genus 1 contains at least one special point.

Remark 1.2. The stability condition from definition 1.1 is equivalent to the marked curve having only finitely many automorphisms, see [HM98, p. 47].

In order to define a moduli space of stable curves, we have to introduce a notion of family of stable curves.

Definition 1.3. A scheme S is said to be algebraic if its structure morphism $S \to \operatorname{Spec} k$ is of finite type.

Definition 1.4. Let S be an algebraic scheme. A **family of stable n-pointed curves** over S is given by the data of a flat, projective morphism of schemes $\pi : F \to S$ and n sections $\sigma_1, \ldots, \sigma_n : S \to F$ such that for every $s \in S$ the fibre $(F_s, \sigma_1(s), \ldots, \sigma_n(s))$ is a stable curve.

An **isomorphism** between two families $(F, \pi, \sigma_1, \ldots, \sigma_n)$ and $(G, \pi', \sigma'_1, \ldots, \sigma'_n)$ is given by an isomorphism $\gamma : F \to G$ compatible with the structure, i.e.

$$\begin{array}{ccc} F & \xrightarrow{\gamma} & G \\ {\scriptstyle\pi}\downarrow & \swarrow{\scriptstyle\pi'} & \\ S & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} F & \xrightarrow{\gamma} & G \\ {\scriptstyle\sigma_i}\uparrow & \nearrow{\scriptstyle\sigma'_i} & \\ S & & \end{array}$$

commute.

Consider the induced functor

$$\begin{aligned} \mathcal{M}_{g,n} : \mathfrak{Scheme} &\to \mathfrak{Set} \\ S &\mapsto \big\{(F, \pi, \underline{\sigma}) \text{ family of stable curves}\big\} / \text{ isomorphisms.} \end{aligned}$$

The representing object (if it exists) is the **moduli space of stable n-pointed curves** of genus g and is denoted $\overline{M}_{g,n}$. The locus of smooth curves is denoted $M_{g,n}$.

Sometimes it is customary to index the marks with a set different from $\{1, \ldots, n\}$, say A. In this case we will adapt notation slightly by writing $\overline{M}_{g,A}$ instead.

Of course the moduli spaces just defined do exist – this is a result due to Deligne, Mumford and Knudsen.

Theorem 1.5. *Coarse moduli spaces $\overline{M}_{g,n}$ exist as projective varieties ([HM98, theorem 2.15]).*

If $2g - 2 + n > 0$ or equivalently

$$g \geq 2 \vee \big(g = 1 \wedge n \geq 1\big) \vee \big(g = 0 \wedge n \geq 3\big)$$

then $\overline{M}_{g,n}$ is non-empty of dimension $3g - 3 + n$ ([Beh97, proposition 2]).

Remark 1.6. The formula for the dimension of the moduli space of one-pointed elliptic curves yields $\dim \overline{M}_{1,1} = 1$, i.e. there is a one parameter family of such curves. This parameter is commonly referred to as *j-invariant*. Note however that by definition $\overline{M}_{1,0} = \emptyset$ despite the expression for the dimension seemingly equating to 0.

We now turn towards stable *maps*. They add more structure to the abstract curves used above by introducing a map establishing a relation with some ambient space X.

Definition 1.7. Let X be an algebraic scheme and let $(C, \underline{p})$ be an n-pointed curve of genus g (i.e. projective, connected, reduced, and nodal at worst). Furthermore fix some $\beta \in A_1(X)$. A tuple $(C, \underline{p}, f)$ where $f : C \to X$ is a morphism is called **stable map of degree** β if and only if

1. $f_*[C] = \beta$ and
2. for every twig $D \subseteq C$ with $f_*[D] = 0$, the twig is stable as marked curve, that is

$$\big(g(D) \geq 2\big) \vee \big(g(D) = 1 \wedge s \geq 1\big) \vee \big(g(D) = 0 \wedge s \geq 3\big),$$

   where s is the number of special points on D. A twig which is mapped to a point is said to be **contracted**.

If C and the marks are clear from context we denote the stable map simply by f again.

An **automorphism** of a stable map is an element $\gamma \in \operatorname{Aut}(C)$ which respects the marks in the sense of definition 1.1 and the map f in the sense of the diagram

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & C \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle f} & \\ X & & \end{array}$$

commuting.

A stable map is said to be **irreducible** or **smooth** if its domain curve is irreducible or smooth respectively.

Remark 1.8. The stability condition from the above definition is equivalent to the group of automorphisms of f being finite ([Gat03b, remark 1.1.10]).

Again, eventually we want to introduce a moduli space of stable maps and therefore need a concept of family.

Definition 1.9. Fix some algebraic schemes X and S. A **family of stable maps of n-pointed curves of genus** g is a tuple $(\pi : F \to S, \sigma_1, \ldots, \sigma_n, \mu)$ where $\pi : F \to S$, $\mu : F \to X$ are morphisms of schemes and $\sigma_i : S \to F$ are sections subject to the condition

$$\forall s \in S : \qquad \big(F_s, \sigma_1(s), \ldots, \sigma_n(s), \mu|_{F_s}\big) \text{ is a stable map.}$$

Two families $(\pi : F \to S, \sigma_1, \dots, \sigma_n, \mu)$ and $(\pi' : G \to S, \sigma'_1, \dots, \sigma'_n, \mu')$ over the same base scheme are **isomorphic** if there exists an isomorphism $\gamma : F \to G$ such that all of the following diagrams commute.

$$\begin{array}{ccc} F & \xrightarrow{\gamma} & G \\ {\scriptstyle\pi}\downarrow & \swarrow{\scriptstyle\pi'} & \\ S & & \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\gamma} & G \\ {\scriptstyle\sigma_i}\uparrow & \nearrow{\scriptstyle\sigma'_i} & \\ S & & \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\gamma} & G \\ {\scriptstyle\mu}\downarrow & \swarrow{\scriptstyle\mu'} & \\ X & & \end{array}$$

Again there is a functor

$$\mathcal{M}_{g,n}(X, \beta) : \mathfrak{Scheme} \to \mathfrak{Set}$$
$$S \mapsto \left\{ \begin{array}{l} (\pi : F \to S, \underline{\sigma}, \mu) \\ \text{family of stable maps} \end{array} \right\} \Big/ \text{ isomorphism}$$

whose representing object (if it exists) is denoted by $\overline{M}_{g,n}(X, \beta)$ and called **moduli space of stable maps of n-pointed curved of degree** β. The locus of smooth maps is denoted by $M_{g,n}(X, \beta)$. Their difference $\overline{M}_{g,n}(X, \beta) \setminus M_{g,n}(X, \beta)$ is called **boundary**.

The following existence result can be found in [FP95, Theorem 1].

Theorem 1.10. *Let X be a projective, algebraic scheme. Then $\overline{M}_{g,n}(X, \beta)$ exists as a projective coarse moduli space.*

Remark 1.11. Recall that $A_1(\mathbb{P}^r) = \mathbb{Z}$ for any $r \geq 1$. Thus, if we are looking at the case $X = \mathbb{P}^r$ we may just as well write $\overline{M}_{g,n}(\mathbb{P}^r, d)$ for some integer d rather than having $\beta = d[\text{line}] \in A_1(\mathbb{P}^r)$.

Definition 1.12. A variety X is called **convex** if

$$\forall \text{ maps } \mu : \mathbb{P}^1 \to X : \qquad H^1(\mathbb{P}^1, \mu^* T_X) = 0.$$

Lemma 1.13. *Projective spaces $\mathbb{P}^r$ are convex for all r.*

Proof. Set $X := \mathbb{P}^r$ and let $\mu : \mathbb{P}^1 \to X$ be some morphism. Consider the Euler sequence ([Gat03a, 7.4.15])

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{\oplus(r+1)} \to T_X \to 0.$$

and apply the pull-back μ^*. This results in another short exact sequence since X is smooth and thus all sheaves of modules in the Euler sequence are vector bundles. We obtain

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mu^*\big(\mathcal{O}_X(1)\big)^{\oplus(r+1)} \to \mu^* T_X \to 0. \tag{1.1}$$

The term in the middle may be rewritten: $\mu^*\mathcal{O}_X(1)$ is a line bundle on $\mathbb{P}^1$ and therefore can be expressed as $\mathcal{O}_{\mathbb{P}^1}(z)$ for some $z \in \mathbb{Z}$. The integer is given as the degree of μ and therefore non-negative. But now consider part of the long exact cohomology sequence induced by (1.1):

$$\cdots \to \underbrace{H^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(z)\big)}_{\cong H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2-z)\right) = 0}{}^{\oplus(r+1)} \to H^1(\mathbb{P}^1, \mu^* T_X) \to \underbrace{0.}_{\text{since } \dim \mathbb{P}^1 = 1}$$

This proves the claim. ∎

In [FP95, Theorem 2] we find more information on the rational case:

Theorem 1.14. *Let X be a projective, smooth, convex variety. Then*

1. *$\overline{M}_{0,n}(X,\beta)$ is a normal, projective variety of pure dimension*

$$\dim X + \int_\beta c_1(T_X) + n - 3.$$

2. *$\overline{M}_{0,n}(X,\beta)$ is locally a quotient of a non-singular variety by a finite group.*

In particular, for $X = \mathbb{P}^r$ recall $c_1(T_X) = H^{r+1}$ and thus we get

$$\dim \overline{M}_{0,n}(\mathbb{P}^r, d) = rd + r + d + n - 3.$$

Remark 1.15. If $\overline{M}_{g,n}$ exists then $\overline{M}_{g,n}(X,0) \cong \overline{M}_{g,n} \times X$, see [BM96, section 7, property 1] and

$$\dim \overline{M}_{g,n+m}(X,\beta) = \dim \overline{M}_{g,n}(X,\beta) + m.$$

Remark 1.16. The modern theory of moduli spaces is formulated using the language of stacks rather than schemes and varieties. Therefore, it would be most accurate to use the term *moduli stack* rather than space. Generally speaking one of the benefits of stacks is that they provide a more refined notion of quotient space by allowing individual points of a stack to have non-trivial automorphism groups. This already suggests why stacks are better suited to describe moduli spaces: the "points" in a moduli space are geometric objects themselves and as such may admit non-trivial automorphisms. The benefits of stacks however come with the disadvantage of heavy technical machinery needed to only define these objects, see [Beh14]. Given the scope of this thesis we will not pursue this track.

1.2 The Boundary

By definition any element in the boundary of the compactified moduli space $\overline{M}_{1,n}(X,\beta)$ is a map defined on a reducible source curve C of genus 1. Obviously these are not the type of curves we had in mind when we asked our enumerative question on the number of genus 1 curves. But simply removing them and working on $M_{1,n}(X,\beta)$ is not an option either: the locus of smooth curves is not compact and thus the entire machinery of intersection theory would not be available. Therefore we will keep them but make it the central issue of this thesis to determine the influence these unwanted curves may have on our enumerative counts. In this section we start by describing the combinatoric structure of boundary curves.

Remark 1.17. The term "boundary" might be a bit misleading because it might suggest that $M = M_{1,n}(X,\beta)$ is dense in $\overline{M}$ which is not the case – in fact we will see in chapter 2 that $\overline{M}$ may even be considerably larger than M. Thus a much more appropriate formulation would be: "$\overline{M}$ is a compact space containing M and it arose by adding only geometrically meaningful points". This longish phrase indicates the definition of $\overline{M}$ we use here is neither canonical nor unique, but geometrically motivated. Additionally it is a particularly successful one, see [PT14].

The notation comes with yet another disadvantage: at times it might be necessary to speak about the topological closure on M in $\overline{M}$ but the canonical symbol for this is obviously already in use. Therefore we use the clumsier notation $cl(M)$ to denote the closure when needed.

Construction 1.18. Boundary components of $\overline{M}_{1,n}(X,\beta)$ are characterized by the combinatoric structure of the source curve of a general element. By "combinatoric structure" we mean the number of twigs of various genera and how they are attached to each other. An element is said to be "general" if all elements of that type constitute a dense, open set. Intuitively, if we were to pick a random element from a component of $\overline{M}$, we would get a general element with probability 1.

In order to visualize such general elements and thereby a component of the moduli space we will use a suggestive pictogram notation. Genus 0 twigs will always be drawn as simple lines, genus 1 twigs will always contain a loop, see 1.1. This depiction of elliptic twigs might be confused with a rational twig glued onto itself forming a singular curve. However the later situation is not enumeratively relevant as we will see in corollary 2.14. Also twigs of higher genus are not relevant as we will see in corollary 1.22. Marks are represented by dots and – if relevant – the degree with which the twig is mapped to X is written next to it. Please note that a pictogram always represents an abstract curve and not its image under f in X.

(a) Genus 0 curve. (b) Smooth genus 1 curve.

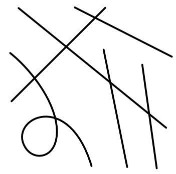
Figure 1.1: Elementary components of pictogram notation.

In the next definition we construct a graph that captures the combinatoric structure of a curve. This is a simplification of the construction performed in [BM96].

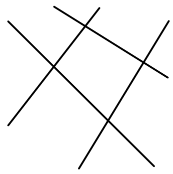
Definition 1.19. Let C be a connected curve with double points as worst singularities. Define an undirected graph $G = (V, E)$ with vertices $V = \{C_1, \dots, C_N\}$ the twigs of C, and edges $\{C_i, C_j\}$ for every singularity of C in which C_i and C_j meet transversally. Note that this definition explicitly allows for parallel edges and self loops in G. We say that C is a **tree** if and only if G is a tree.

Example 1.20. See figure 1.2.

The next lemma expresses the genus of a nodal curve in terms of the genera of its twigs, see [Gat03b, 1.1.2] or [HM98, p. 48].



(a) A tree.



(b) Not a tree.

Figure 1.2: Example of a curve which is a tree, and one which is not.

Lemma 1.21. *Let C be a curve with twigs $C_1, \dots, C_N$ which are glued at s points. The genus of C is given by the formula*

$$g = s - N + 1 + \sum_{i=1}^{N} g(C_i) = s + 1 + \sum_{i=1}^{N} \big(g(C_i) - 1\big).$$

In case of elliptic curves, i.e. $g = 1$ the lemma yields:

Corollary 1.22. *Let C be a nodal genus 1 curve. Keeping notation from lemma 1.21 it holds that either*

1. *C is a tree, $s = N - 1$ and it contains exactly one twig of genus 1 (the remaining twigs have genus 0) or*
2. *the graph constructed in definition 1.19 contains exactly one cycle (or self loop), $s = N$ and all twigs are of genus 0.*

In particular twigs of genus larger than 1 do not occur.

Proof. Using $g = 1$ in lemma 1.21 gives

$$s = \sum_{i=1}^{N} \big(\underbrace{1 - g(C_i)}_{\leq 1}\big) \geq N$$

and C being connected implies $s \geq N - 1$. The two possible values for s give rise to the different situations of the claim. □

In particular the curves from figure 1.2 are both of genus 1.

1.3 Gromov-Witten Invariants

We will now introduce Gromov-Witten invariants and thereby establish the language needed to re-formulate Getzler's theorem (theorem 1.28). In order to get started we introduce some natural morphisms on the moduli space giving it a geometric structure.

Construction 1.23. Let $(C, \underline{p})$ be an n-marked curve of genus g which is not stable. If C satisfies $2g - 2 + n > 0$ the failure to being stable is caused by at least one rational twig containing less than three special points. Let C_j be an unstable, rational twig. We construct a new curve $(\widetilde{C}, \underline{p})$ by removing C_j and then reattach any marks or further twigs formerly attached to C_j to C at the point $C \cap C_j$. Geometrically one can imagine that C_j has been contracted to a point. If this operation is iteratively performed for every unstable twig then we will obtain a stable curve $[C]^{stab}$ (the condition $2g - 2 + n > 0$ ensures that the process terminates with at least one twig left). The curve $[C]^{stab}$ is called **stabilization** of C, compare [BM96, proposition 1.13]. The same construction can be carried out to stabilize maps.

Definition 1.24. a) Define for $i = 1, \ldots, n$ the **evaluation map**

$$\begin{aligned} ev_i : \overline{M}_{g,n}(X, \beta) &\to X, \\ f &\mapsto f(p_i). \end{aligned}$$

b) For any p_i there is a **forgetful morphism**

$$\begin{aligned} \pi_i : \overline{M}_{g,n}(X, \beta) &\to \overline{M}_{g,\{1,\ldots,i-1,i+1,\ldots,n\}}(X, \beta) \\ (C, p_1, \ldots, p_n, f) &\mapsto [(C, p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n, f)]^{stab} \end{aligned}$$

and analogously for stable curves. In either case π_n is the universal curve (for stable curves see [Beh97], for stable maps see [BM96, corollary 4.6]).

c) There also exists a forgetful morphism which forgets the map to X:

$$\begin{aligned} \rho : \overline{M}_{g,n}(X, \beta) &\to \overline{M}_{g,n} \\ (C, \underline{p}, f) &\mapsto [(C, \underline{p})]^{stab}. \end{aligned}$$

All of these maps are indeed morphisms of schemes (e.g. for evaluation maps see [BM96, proposition 5.5]).

The evaluation maps from definition 1.24 are the key to approaching enumerative questions by intersection theory. Let geometric objects, i.e. subvarieties $B_1, \ldots, B_n \subseteq X$ like points and lines be given. Then each of these objects gives rise to a class $[B_i] \in A^*(X)$ which may then be pulled back along ev_i to give $ev_i^*[B_i] \in A^*(\overline{M}_{g,n}(X, \beta))$. Geometrically the support of a cycle representation of such a class should consist of all maps f whose image in X is a curve running through B_i. Therefore we want to intersect all these pull-backs in order to satisfy all incidence conditions at once. If n and the objects B_i were chosen appropriately the intersection

$$ev_1^*[B_1] \cdots ev_n^*[B_n]$$

should be a co-cycle in dimension 0 and therefore intersecting with the fundamental class followed by a degree evaluation should give us the answer to our enumerative problem.

However, things are not going as smoothly as one could hope. Geometrically we are interested in smooth elliptic curves and thus we want to choose the conditions in a way that their codimensions sum up to $\dim M_{1,n}(X, \beta)$. The problem lies in the boundary: there are components with dimension strictly larger than the smooth locus – they are determined in the next chapter. As a consequence, the procedure just described fails.

In order to remedy this problem Behrend and Fantechi [BF97] introduced the concept of *virtual* or *expected dimension* and the so-called *virtual fundamental* *class.* The idea is that $\operatorname{vdim} \overline{M}$ is the number which one would expect for the dimension of $\overline{M}$ – we will see in theorem 2.12 that in the special case of elliptic curves in $\mathbb{P}^r$ it is in fact equal to $\dim M$. The virtual fundamental class is then a class $[\overline{M}]^{virt} \in A_{\operatorname{vdim} \overline{M}}(\overline{M})$ and replaces the usual fundamental class. Of course Gromov-Witten invariants are defined using this class:

Definition 1.25. Let $\gamma_1, \dots, \gamma_n \in A^*(X)$. The number

$$\langle \gamma_1 \cdots \gamma_n \rangle_{g,\beta} := \deg \left(ev_1^* \gamma_1 \cdots ev_n^* \gamma_n \cdot \left[\overline{M}_{g,n}(X, \beta) \right]^{virt} \right)$$

is called **Gromov-Witten invariant** (compare [Gat03b]). Recall that deg denotes the degree of the dimension 0 part. Another common notation is

$$\langle \gamma_1 \cdots \gamma_n \rangle_{g,n} = \int_{[\overline{M}_{g,n}(X,\beta)]^{virt}} ev_1^* \gamma_1 \cdots ev_n^* \gamma_n$$

and correspondingly one speaks of "integrating" the cycle $ev_1^* \gamma_1 \cdots ev_n^* \gamma_n$ over $[\overline{M}_{g,n}(X, \beta)]^{virt}$.

At this point we are obviously lacking the definition of virtual dimension and virtual fundamental classes. The reason why we do not simply state it here lies in its sheer technical complexity. Both terms are defined using the language of stacks and need concepts such as perfect obstruction theories and intrinsic normal cones. This however is far beyond the scope of this thesis and thus we have to content ourselves by remarking the following properties:

1. vdim and $[-]^{virt}$ can be defined for any separated Deligne-Mumford stack admitting a perfect obstruction theory ([BF97]). In particular the concept applies to more general situations than that of $\overline{M}_{g,n}(X, \beta)$.

2. For genus 0 and convex X the virtual fundamental class is given by the usual fundamental class $[\overline{M}_{0,n}(X, d)]^{virt} = [\overline{M}_{0,n}(X, d)]$, [BM96, theorem 7.5].

3. $\operatorname{vdim} \overline{M}_{g,n}(X, \beta) = -K_X \beta + (\dim X - 3)(g - 1) + n$, [CK99, p. 175].

4. The construction of the virtual fundamental class is local.

Remark 1.26 (number of conditions). In order for Gromov-Witten invariants to be non-zero, we have to ensure that the intersection does have non-trivial dimension 0 part. This is equivalent to

$$\sum \operatorname{codim}_X \gamma_i = \operatorname{vdim} \overline{M}_{g,n}(X, \beta).$$

Since we are interested in elliptic Gromov-Witten invariants in $\mathbb{P}^3$, this specializes as $X = \mathbb{P}^3$ and $g = 1$. Furthermore, in order to implement point and line incidence conditions, the γ_i will be taken to be the class of a point or line. The codimensions are obviously 3 and 2 respectively. If we call a the number of line incidence conditions and b the number of point incidences we want to implement (and thus $n = a + b$) we get

$$\begin{aligned} 3b + 2a &= 4d + (a + b) \\ \iff 2b + a &= 4d. \end{aligned}$$

This tells us how many conditions we have to impose (compare [Get97]).

Definition 1.27. We will use the following simplified notation for rational and elliptic Gromov-Witten invariants with respect to a lines and b points:

$$N_{ab}^{(g)} := \langle \gamma_1 \cdots \gamma_{a+b} \rangle_{g,(2b+a)/4}.$$

Obviously this only makes sense if $2b + a$ is divisible by 4.

The following theorem was postulated by Getzler in [Get97, theorem 6.1] and is the main theorem of this thesis. We will prove it in chapter 3.

Theorem 1.28. *The number of elliptic curves in $\mathbb{P}^3$ of degree d passing through a number of a generic lines and b generic points, where $4d = a + 2b$, equals*

$$N_{ab}^{(1)} + \frac{2d-1}{12} N_{ab}^{(0)}.$$

Example 1.29. Table 1.1 gives some values for rational and elliptic Gromov-Witten invariants depending on the values for a and b as well as Getzler's claim for the corresponding number of curves. The table has been taken from [Get97, Table 1].

1.4 Tautological Classes on $\overline{M}_{g,n}$ and $\overline{M}_{g,n}(X, \beta)$

In this section we want to introduce the so-called psi- and lambda-classes on the moduli spaces of stable curves and maps. Furthermore, we will quote some results that allow for evaluation of these classes.

Construction 1.30 (psi-classes). Consider an element $(C, \underline{p}) \in \overline{M}_{g,n}$ and fix an index i. As C is one-dimensional and p_i a smooth point of C, the cotangent space of C at p_i is one-dimensional as well. If we now allow the point $(C, \underline{p})$ to vary inside the moduli space this gives rise to a line bundle. More formally, consider the relative cotangent sheaf $\Omega_{\overline{M}_{g,n+1}/\overline{M}_{g,n}}$ of the universal curve

$$\overline{M}_{g,n+1} \to \overline{M}_{g,n}.$$

Then the pull-back along the section σ_{n+1} corresponding to the forgotten $(n+1)$-th mark is denoted $\mathbb{T}_i$. This is precisely the desired line bundle. Its first Chern class is denoted $\psi_i := c_1(\mathbb{T}_i) \in A^1(\overline{M}_{g,n})$ and referred to as i-th **psi-class**. The same construction is used in the case of stable maps as well.

Lemma 1.31.

$$\int_{[\overline{M}_{1,1}]} \psi_1 = \frac{1}{24}.$$

Proof. [Wit91, equation 2.46]. □

Lemma 1.32. *For the forgetful morphism $\pi : \overline{M}_{0,n+1}(X, d) \to \overline{M}_{0,n}(X, d)$ we have*

$$\forall i = 1, \ldots, n : \quad \pi_* \psi_i = (2g - 2 + n)[\overline{M}_{0,n}(X, d)].$$

Proof. Special case of [Gat03b, corollary 1.3.5]. □

Construction 1.33 (lambda-classes). Consider again the universal curve

$$\pi : \underbrace{\overline{M}_{g,n+1}}_{=:C} \to \underbrace{\overline{M}_{g,n}}_{=:M}$$

and its relative dualizing sheaf $\omega_{C/M}$. Define the **Hodge bundle** $\mathbb{E} := \pi_* \omega_{C/M}$. Its fibre at a point $(C, \underline{p})$ is given by $H^0(C, \omega_C)$, where ω_C is the dualizing sheaf of C. Thus it is a rank g vector bundle and we may consider the Chern classes

$$\lambda_i := c_i(\mathbb{E}) \in A^1(\overline{M}_{g,n})$$

for $i = 1, \dots, g$. They are called **lambda-classes**, compare [HM98, p. 155 – 156]. In the case of $g = 1$ we will abbreviate notation by writing $\lambda = \lambda_1$.

Theorem 1.34. *[FP03, Theorem 1]:*

$$\int_{[\overline{M}_{g,n}]} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \lambda_g = \binom{2g+n-3}{\alpha_1, \dots, \alpha_n} \int_{[\overline{M}_{g,1}]} \psi_1^{2g-2} \lambda_g$$

and

$$\int_{[\overline{M}_{g,1}]} \psi_1^{2g-2} \lambda_g = \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!},$$

where B_n denotes the n-th Bernoulli number.

Put $g = 1$ in the above theorem to obtain:

Corollary 1.35.

$$\int_{[\overline{M}_{1,1}]} \lambda = \frac{1}{24}$$

Note that $\overline{M}_{1,1} \cong \mathbb{P}^1$, hence $\lambda = c[\text{point}]$ for some $c \in \mathbb{Q}$. This coefficient is precisely given by the degree evaluation, i.e. $c = 1/24$.

Construction 1.36. Let H be a hyperplane in $X = \mathbb{P}^r$ and consider the evaluation map

$$ev_{n+1} : \overline{M}_{0,n+1}(X, d) \to X.$$

The pull-back $ev_{n+1}^* H \in A^1(\overline{M}_{0,n+1}(X, d))$ has codimension 1 because the hyperplane enforces exactly one condition.

Lemma 1.37. *Let $\pi : \overline{M}_{0,n+1}(X, d) \to \overline{M}_{0,n}(X, d)$ be the forgetful morphism which drops the $(n+1)$-st point. Then*

$$\pi_* ev_{n+1}^* H = d[\overline{M}_{0,n}(X, d)].$$

Proof. Special case of [Gat03b, corollary 1.3.4]. □

d	(a,b)	$N_{ab}^{(0)}$	$N_{ab}^{(1)}$	$N_{ab}^{(1)} + \frac{2d-1}{12} N_{ab}^{(0)}$
1	(0, 2)	1	-1/12	0
	(2, 1)	1	-1/12	0
	(4, 0)	2	-1/6	0
2	(0, 4)	0	0	0
	(2, 3)	1	-1/4	0
	(4, 2)	4	-1	0
	(6, 1)	18	-4 1/2	0
	(8, 0)	92	-23	0
3	(0, 6)	1	-5/12	0
	(2, 5)	5	-2 1/12	0
	(4, 4)	30	-12 1/2	0
	(6, 3)	190	-78 1/6	1
	(8, 2)	1312	-532 2/3	14
	(10, 1)	9864	-3960	150
	(12, 0)	80160	-31900	1500
4	(0, 8)	4	-1 1/3	1
	(2, 7)	58	-29 5/6	4
	(4, 6)	480	-248	32
	(6, 5)	4000	-2023 1/3	310
	(8, 4)	35104	-17257 1/3	3220
	(10, 3)	327888	-156594	34674
	(12, 2)	3259680	-1515824	385656
	(14, 1)	34382544	-15620216	4436268
	(16, 0)	383306880	-170763640	52832040
5	(0, 10)	105	-36 3/4	42
	(2, 9)	1265	-594 3/4	354
	(4, 8)	13354	-6523 1/2	3492
	(6, 7)	139098	-66274 1/2	38049
	(8, 6)	1492616	-677808	441654
	(10, 5)	16744080	-7179606	5378454
	(12, 4)	197240400	-79637976	68292324
	(14, 3)	2440235712	-928521900	901654884
	(16, 2)	31658432256	-11385660384	12358163808
	(18, 1)	429750191232	-146713008096	175599635328
	(20, 0)	6089786376960	-1984020394752	2583319387968

Table 1.1: Some example values for Gromov-Witten invariants and the claimed number of elliptic curves. Notation as in theorem 1.28.

Chapter 2

Dimension

In chapter 1 we introduced the moduli space of stable maps and already hinted at the odd behaviour of the boundary. In this chapter we develop a calculus allowing us to compute the dimension of basically any irreducible component of $\overline{M}_{1,n}(\mathbb{P}^r, d)$. We then identify the components of excessive dimension in corollary 2.14.

From now on the ambient space X will always be $\mathbb{P}^r$ and we restrict ourselves to moduli spaces of elliptic curves!

The following lemma relates the gluing of curves to fibre products of subschemes of suitable moduli spaces. It is a simple application of properties that could be stated much more generally.

Lemma 2.1. *Let $D \subseteq \overline{M}_{g,n+1}(X, d)$ and $D' \subseteq \overline{M}_{g',n'+1}(X, d')$ be irreducible components of their respective moduli spaces. Let $(C, \underline{p}, f)$ and $(C', \underline{p}', f')$ be general elements of D and D' respectively. If $G \subseteq \overline{M}_{g+g',n+n'}(X, d+d')$ denotes the irreducible component with general element given by gluing C and C' at the $(n + 1)$ and $(n' + 1)$-st mark then there is a fibre product diagram*

$$\begin{array}{ccc} G & \longrightarrow & D \\ \downarrow & \square & \downarrow{\scriptstyle ev_{n+1}} \\ D' & \xrightarrow{ev_{n'+1}} & X. \end{array}$$

Proof. By [BM96, section 7, property 2 and 3] there is a fibre product diagram

$$\begin{array}{ccc} G & \longrightarrow & X \\ \downarrow & \square & \downarrow{\scriptstyle \Delta} \\ D \times D' & \xrightarrow{(ev_{n+1}, ev_{n'+1})} & X \times X. \end{array}$$

It is now easy to construct an isomorphism $G \cong D' \times_X D$ using the universal property of both fibre products. □

In order to turn lemma 2.1 into a dimension formula we use the following general fact:

Lemma 2.2. *Let $f : A \to C$ and $g : B \to C$ be flat, surjective morphisms of schemes with C irreducible and A and B pure dimensional. Then*

$$\dim(A \times_C B) = \dim A + \dim B - \dim C.$$

Proof. Let $x \in C$ be any point. By [Liu02, corollary 4.3.14] it holds

$$\dim f^{-1}(x) = \dim A - \dim C \text{ and}$$
$$\dim g^{-1}(x) = \dim B - \dim C.$$

We check that all assumptions of [Liu02, corollary 4.3.14] are satisfied for $h : A \times_C B \to C$ as well:

- the morphism is flat and surjective as both properties are stable under base change and composition and
- the fibre product of pure dimensional schemes over irreducible C is pure dimensional.

Hence by the afore mentioned corollary

$$\dim h^{-1}(x) = \dim(A \times_C B) - \dim C.$$

Furthermore, $x = x \times_C x = x \times x$ and thus we may write

$$h^{-1}(x) = A \times_C B \times_C x = (A \times_C x) \times (x \times_C B) = f^{-1}(x) \times g^{-1}(x)$$

and the claim follows from the dimension formula for products:

$$\dim\big(f^{-1}(x) \times g^{-1}(x)\big) = \dim f^{-1}(x) + \dim g^{-1}(x). \qquad \square$$

Of course we want to combine the fibre product structure from lemma 2.1 with the dimension formula from lemma 2.2. For this we need to check that the evaluation maps are flat and surjective. For flatness see [KV03, lemma 2.5.1] (the argument can be applied in positive genus without any change) and for surjectivity use the next lemma.

Lemma 2.3. *Let $d > 0$ and $I \subseteq \{1, \dots, n\}$. We define the combined evaluation map on the topological closure of the smooth locus*

$$ev = (ev_i)_{i \in I} : cl\big(M_{1,n}(X, d)\big) \to X^{|I|}.$$

If $|I| < r + 1$ then ev is surjective.

Proof. The group of linear automorphisms of k^{r+1} acts $(r+1)$-transitively on the set of linearly independent tuples of vectors in k^{r+1}. In particular, if $m := |I| \le r + 1$ it acts m-transitively and we define

$$U_m := \big\{(b_1, \dots, b_m) \in (k^{r+1})^m \;\big|\; \dim\langle b_1, \dots, b_m\rangle_k = m\big\}.$$

A tuple lying in U_m amounts to the points being in general position. Passing to projective space we get that the group of projective linear automorphisms of X acts m-transitively on the set of m-tuples in general position. Let U_m^P denote this set.

Now let $f : C \to X$ be a general element of $M_{1,n}(X, d)$. Generality means precisely that $x := ev(f)$ is an m-tuple in general position. Hence for any other $y \in U_m^P$ there is a linear automorphism α of X such that $\alpha(x) = y$. Note that $\alpha \circ f \in M_{1,n}(X, d)$ because $\deg \alpha = 1$ and thus it does not change the degree of any twig. All together this shows that ev maps surjectively to the dense open set U_m^P. Hence by continuity and compactness of $cl(M_{1,n}(X, d))$ the claim follows. □

The following theorem gives a useful formula for dimensions of boundary components:

Theorem 2.4. *Let D be a component of $\overline{M}_{1,n}(X, d)$ and $f : (C, \underline{p}) \to X$ be a general element of D. Let $C_1, \dots, C_N$ denote the twigs of C and say that C_i has genus g_i, is mapped with degree d_i and contains n_i of the n marks. Moreover let s_i denote the number of half-edges incident to C_i in the graph defined in definition 1.19 (i.e. self-loops are counted twice) and finally let s be the total number of gluing points. Then*

$$\dim D = \sum_{i=1}^{N} \dim M_{g_i, n_i + s_i}(X, d_i) - s \dim X.$$

In particular, if all of the spaces occurring exist then by remark 1.15

$$\dim D = \sum_{i=1}^{N} \dim M_{g_i, n_i}(X, d_i) - s(\dim X - 2).$$

Proof. We proceed by induction on N. Let $N = 1$ and consider the cases:

1. $s = 0$ and thus C is irreducible and smooth: In this case we have $D = cl(M_{1,n}(X, d))$ and there is nothing to show.
2. C is of genus 0 and glued to itself in a single point: Then D is the space of genus 0 curves with $n + 2$ marks such that $ev_{n+1} = ev_{n+2}$. This space is precisely the pull back of the diagonal $\Delta \subseteq X \times_k X$ along the morphism

   $$(ev_{n+1}, ev_{n+2}) : \overline{M}_{0,n+2}(X, d) \to X \times_k X,$$

   i.e. there is a cartesian diagram

   $$\begin{array}{ccc} D & \longrightarrow & X \\ \big\downarrow & \square & \big\downarrow \Delta \\ \overline{M}_{0,n+2}(X, d) & \xrightarrow{(ev_{n+1}, ev_{n+2})} & X \times X \end{array}$$

   and we get $\dim D = \dim \overline{M}_{0,n+2}(X, d) - \dim X$.

Now assume $N > 1$ and without loss of generality let C_1 be a twig of C such that $C' := C_2 \cup \dots \cup C_N$ is still connected. Furthermore, if we add new marks at points where C_1 would be attached to C', it is again a stable map and thus an element in $\overline{M}_{1-g_1, n-n_1+s_1}(X, d - d_1)$. Since C' consists of only $N - 1$ twigs, induction hypothesis may be applied to the irreducible component D' with general element C'.

Again we distinguish two cases:

1. C_1 is glued to C' in only one point. In this case $M_{g_1,n_1+1}(X,d_1)\times_X D'$ is a dense, open subset of D and thus

$$\dim D = \dim M_{g_1,n_1+1}(X,d_1) - \dim X + \dim D'.$$

2. C_1 is glued to the rest of C in 2 points. Analogously we have that $M_{g_1,n_1+2}(X,d_1)\times_{X\times X} D'$ is a dense, open subset of D and thus

$$\dim D = \dim M_{g_1,n_1+2}(X,d_1) - 2\dim X + \dim D'.$$

In either case the claim follows by application of the induction hypothesis to D'. □

Note that theorem 2.4 essentially reduced the task of computing the dimension of the compactified moduli space $\overline{M}_{1,n}(X,d)$ to the computation of $\dim M_{1,n}(X,d)$: the dimension of the compactified moduli space is the maximum of the dimensions of the individual components. The theorem expresses the latter with respect to the dimension of not compactified moduli spaces.

We will now introduce a valuable tool for further investigation: let $f : C \to X$ be a stable map and set $P := \sum p_i$ the divisor of all marks. Then there is a long exact sequence of hyper-Ext groups [CK99]:

$$\begin{aligned}\cdots \to \operatorname{Ext}^i\big(f^*\Omega_X \to \Omega_C(P), \mathcal{O}_C\big) &\to \operatorname{Ext}^i\big(\Omega_C(P), \mathcal{O}_C\big)\\ \to \operatorname{Ext}^i\big(f^*\Omega_X, \mathcal{O}_C\big) &\to \operatorname{Ext}^{i+1}\big(f^*\Omega_X \to \Omega_C(P), \mathcal{O}_C\big) \to \cdots \qquad (2.1)\end{aligned}$$

Remark 2.5. Sequence (2.1) in the form presented here is an exact sequence of k-modules for ever stable map f. Indeed there is an underlying exact sequence of sheaves on the moduli space $\overline{M}_{1,n}(X,d)$ such that (2.1) is just the fibre at f. This should be kept in mind because later on we will need this global view point. Until then it is important to keep any manipulations of the objects involved natural such that they lift to operations on the underlying sheaves.

We will now present some simplifications of the terms occurring in sequence (2.1). For a start we define the short hand notation

$$E^j := \operatorname{Ext}^j\big(f^*\Omega_X \to \Omega_C(P), \mathcal{O}_C\big).$$

Furthermore we employ smoothness of X and some basic properties of the Ext-functor (see appendix A.3 for details) to obtain:

$$\forall j : \qquad \operatorname{Ext}^j\big(f^*\Omega_X, \mathcal{O}_C\big) \cong \operatorname{Ext}^j(\mathcal{O}_C, f^*T_X) \cong H^j\big(C, f^*T_X\big).$$

This already shows that the hyper-Ext sequence ends after $\operatorname{Ext}^2(\Omega_C(P), \mathcal{O}_C)$.

Remark 2.6. Before we simply further, some geometric explanations are in order:

1. E^0 is the space of infinitesimal automorphisms of f. It is always trivial because f is stable, compare [CK99, p. 175].

2. $\operatorname{Ext}^0(\Omega_C(P), \mathcal{O}_C)$ are the infinitesimal automorphisms of C. In case C is stable as a curve, this space is trivial as well. But in general this need not be the case, see [CK99, p. 175].

3. $H^0(C, f^*T_X)$ are global sections of T_X pulled back to C. The space thus represents the ways in which the image of f in X may be deformed, or equivalently, the deformations of f with C being fixed, see [HM98, p. 96] or [CK99, p. 175].

4. E^1 gives the deformations of f as stable map, meaning deformations of the curve and the map combined. It is therefore isomorphic to the tangent space of $\overline{M}_{1,n}(X, d)$ at f (see [CK99, p. 175]). If the moduli space was smooth, the dimension of this term would be independent of f and equal to the dimension of the moduli space. However for genus 1 this is not the case: there is a singular locus on which $\dim E^1$ exceeds the dimension of the moduli space.

5. $\mathrm{Ext}^1(\Omega_C(P), \mathcal{O}_C)$ is the space of first-order deformations of $(C, \underline{p})$, [HM98, p. 99]. It is depends only on the curve and its marks, not on the map.

6. If the term $H^1(C, f^*T_X)$ vanishes, it follows that the map in front of it is surjective and thus every deformation of C lifts to a deformation of f. In other words, the tangent space E^1 is indeed comprised of combinations of deformations of C and deformations of f which are not due to infinitesimal automorphisms of C.

7. Finally E^2 is an obstruction term and the last non-trivial term in sequence (2.1) as we shall see now.

For smooth C the cotangent sheaf Ω_C is a line bundle and thus we may again apply basic properties of Ext. The result is

$$\forall j: \qquad \mathrm{Ext}^j\big(\Omega_C(P), \mathcal{O}_C\big) \cong H^j\big(C, T_C(-P)\big) \tag{2.2}$$

which shows that $\mathrm{Ext}^2(\Omega_C(P), \mathcal{O}_C)$ is trivial. For nodal C this procedure cannot be applied, but the result still holds:

Lemma 2.7. *For all stable maps $f : C \to X$ it holds* $\mathrm{Ext}^2(\Omega_C(P), \mathcal{O}_C) = 0$.

Proof. By remark B.5 there is an isomorphism

$$\mathrm{Ext}^2\big(\Omega_C(P), \mathcal{O}_C\big) \cong H^0\big(C, \mathcal{E}xt^2(\Omega_C(P), \mathcal{O}_C)\big) \oplus H^1\big(C, \mathcal{E}xt^1(\Omega_C(P), \mathcal{O}_C)\big).$$

As $\mathcal{E}xt^i$ is a local construction and $\Omega_C(P)$ is locally free away from the nodes of C, we get by proposition A.4 that for $i > 0$ the sheaf $\mathcal{E}xt^i(\Omega_C(P), \mathcal{O}_C)$ is a skyscraper sheaf supported in the nodes C. Skyscraper sheaves are flabby, hence

$$H^1\big(C, \mathcal{E}xt^1(\Omega_C(P), \mathcal{O}_C)\big) = 0.$$

It remains to compute $\mathcal{E}xt^2(\Omega_C(P), \mathcal{O}_C)$, or more precisely, its stalk in a given node ν of C. We know $(\mathcal{E}xt^2_C(\Omega_C(P), \mathcal{O}_C))_\nu = \mathrm{Ext}^2_{\mathcal{O}_{C,\nu}}(\Omega_{C,\nu}, \mathcal{O}_{C,\nu})$ because marks are always smooth points of C, i.e. ν is not contained in the support of P. Locally we may assume without loss of generality

$$C = V(xy) \subseteq \operatorname{Spec} k[x, y]$$

or, put differently: $C = \operatorname{Spec} R$ with $R := k[x, y]/\langle xy \rangle$. We identify ν with $\langle x, y \rangle$ such that we may write $\mathcal{O}_{C,\nu} = R_\nu$ and

$$\Omega_{C,\nu} = \langle \mathrm{d}x, \mathrm{d}y \rangle_{R_\nu} / \langle y\,\mathrm{d}x + x\,\mathrm{d}y \rangle_{R_\nu}.$$

Consider the short exact sequence of R_ν-modules

$$0 \to \langle y\,\mathrm{d}x + x\,\mathrm{d}y\rangle_{R_\nu} \to \langle \mathrm{d}x, \mathrm{d}y\rangle_{R_\nu} \to \Omega_{C,\nu} \to 0 \tag{2.3}$$

and note that the first two elements are free, hence projective, hence their Ext^i-groups over R_ν are trivial for $i > 0$. Plugging this into the long exact Ext sequence induced by (2.3) yields $\forall i \geq 2 : \mathrm{Ext}^i(\Omega_{C,\nu}, R_\nu) = 0$ and hence the claim. □

Summing up all the simplifications made to sequence (2.1) we get:

Corollary 2.8. *For all stable maps $f : C \to X$ there is an exact sequence*

$$\begin{aligned} 0 &\to \mathrm{Ext}^0\big(\Omega_C(P), \mathcal{O}_C\big) \to H^0\big(C, f^*T_X\big) \to E^1 \\ &\to \mathrm{Ext}^1\big(\Omega_C(P), \mathcal{O}_C\big) \to H^1\big(C, f^*T_X\big) \to E^2 \to 0. \end{aligned} \tag{2.4}$$

The hyper-Ext groups in this sequence are related to the idea of a *perfect obstruction theory* which we will not elaborate on. However we may at least formally define the "expected" dimension.

Definition 2.9. We call $\mathrm{vdim}\,\overline{M}_{g,n}(X,d) := \dim E^1 - \dim E^2$ the **virtual** or **expected dimension** of $\overline{M}_{g,n}(X,d)$.

Remark 2.10. A priori it is not clear why this expression is well-defined. The E^i and their dimensions do surely depend on the moduli point f. Nevertheless it can be shown, that the difference in dimension is indeed constant on all of $\overline{M}_{g,n}(X,d)$.

In [CK99, p. 175] we find the virtual dimension's numerical value. Specialized to $X = \mathbb{P}^r$ we obtain

Theorem 2.11. *For all genera g, degrees d and number of marks n it holds*

$$\mathrm{vdim}\,\overline{M}_{g,n}(\mathbb{P}^r, d) = (r+1)d + (r-3)(1-g) + n.$$

We now turn to the only component whose dimension has not yet been computed: the smooth locus. The following theorem fills this gap and thereby justifies the name "expected" dimension.

Theorem 2.12. *For all $d > 0$ it holds*

$$\dim M_{1,n}(X,d) = \mathrm{vdim}\,\overline{M}_{1,n}(X,d)$$

and $M_{1,n}(X,d)$ is smooth.

Proof. We already remarked that E^1 is isomorphic to the tangent space at f. It therefore suffices to show that

1. $\dim E^1$ is independent of f in the smooth locus $M_{1,n}(X,d)$ (this implies smoothness) and
2. $E^2 = 0$ for every smooth f.

The latter is accomplished by showing $H^1\big(C, f^*T_X\big) = 0$ and pointing to sequence (2.4). This is essentially a copy of the proof of lemma 1.13. Start with the Euler sequence for $X = \mathbb{P}^r$

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{\oplus(r+1)} \to T_X \to 0$$

and note that all occurring modules are locally free. Hence applying the pull back along f gives again a short exact sequence

$$0 \to \mathcal{O}_C \to f^*\big(\mathcal{O}_X(1)\big)^{\oplus(r+1)} \to f^*T_X \to 0.$$

As always we investigate the long exact cohomology sequence

$$\cdots \to H^1\big(C, f^*\mathcal{O}_X(1)\big)^{\oplus(r+1)} \to H^1\big(C, f^*T_X\big) \to \underbrace{H^2\big(C, \mathcal{O}_C\big)}_{=0}. \tag{2.5}$$

In order to understand $H^1\big(C, f^*\mathcal{O}_X(1)\big)$ recall that line bundles and divisors of smooth curves are in one-to-one correspondence. Therefore we can write $f^*\mathcal{O}_X(1) = \mathcal{O}_C(D)$ for some divisor $D \in \operatorname{Div} C$. The invertible sheaf $\mathcal{O}_X(1)$ corresponds to the divisor $H \in \operatorname{Div} X$, where H is an arbitrary hyperplane of X. This tells us that

$$D = f^*\big(H \cap f(C)\big) = f^*H \cap C = f^*H,$$

where the middle equality is the projection formula of intersection theory. The cardinality of the intersection $H \cap f(C)$ (counted with multiplicities and therefore written as degree) is given by Bézout's theorem as $\deg\big(H \cap f(C)\big) = \deg f(C) = d$. Thus we have $\deg D = d > 0$ and therefore for curves of genus $g = 1$

$$\begin{aligned} &\deg(K_C - D) < \deg K_C = 2g - 2 = 0 \\ \Longrightarrow &H^1\big(C, f^*\mathcal{O}_X(1)\big) \cong H^1\big(C, \mathcal{O}_C(D)\big) \underset{\text{Serre duality}}{\cong} H^0\big(C, \mathcal{O}_C(K_C - D)\big)^\vee = 0. \end{aligned}$$

Using sequence (2.4) completes the proof of the first claim. Note that we used no property of the moduli point f except for smoothness of the underlying marked curve. Thus the result is independent of f and the second part of the claim is proven as well. ☐

Our dimension calculus for components of the moduli space is now complete and we may start to apply it. The next corollary deals with the phenomenon of components of excessive dimension at which we hinted in remark 1.17. We define

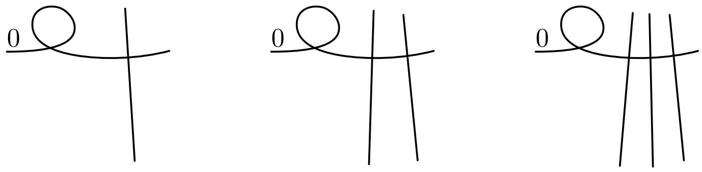
Definition 2.13. An irreducible component $D \subseteq \overline{M}_{g,n}(X,d)$ is said to be of **excessive dimension** and thus relevant to our computations if $\dim D \geq \operatorname{vdim} \overline{M}_{g,n}(X,d)$ and $D \neq cl(M_{g,n}(X,d))$.

The existence of excessive dimensional components might be unexpected and indeed this is the reason why elliptic Gromov-Witten invariants are not enumerative. The following corollary identifies all of them in the case $X = \mathbb{P}^3$ which is claimed in the main theorem.

Corollary 2.14. *An irreducible component* $D \subseteq \overline{M}_{1,n}(\mathbb{P}^3, d)$ *with general element* $f : C \to \mathbb{P}^3$ *is of excessive dimension if and only if* C *is a tree with at most three rational twigs and* f *contracts the genus 1 twig, see figure 2.1. In this case*

$$\dim D = \operatorname{vdim} \overline{M}_{1,n}(\mathbb{P}^3, d) + 3 - s,$$

where s *is the number of rational twigs of* C*.*



(a) Component type D_1. dim = vdim +2. (b) Component type D_2. dim = vdim +1. (c) Component type D_3. dim = vdim.

Figure 2.1: The excessive dimensional components of $\overline{M}_{1,n}(\mathbb{P}^3, d)$ are the ones whose general element is of one of the forms depicted here. In each case there are as many components as there are ways to distribute the degree d to the genus 0 twigs such that no twig is contracted (the components with contracted rational twig lie in the boundary of another component and are thus not relevant). Furthermore this gets multiplied by the possible distributions of the n marks to the twigs.

Proof. Let $f : (C, p_1, \ldots, p_n) \to \mathbb{P}^3$ be a general element of some component $D \subseteq \overline{M}_{1,n}(\mathbb{P}^3, d)$ and let $C_1, \ldots, C_N$ denote the twigs of C. Furthermore let n_i be the number of marked points on C_i, d_i the degree of $f|_{C_i}$ and s_i the number of half edges adjacent to C_i in the graph defined in definition 1.19. In particular,

$$\sum_{i=1}^{N} s_i = 2s.$$

Distinguish two cases based on the results of corollary 1.22:

1. Assume C is not a tree, i.e. all twigs are rational. In this case we plug theorem 1.14 into the formula from theorem 2.4 and compute

$$\begin{aligned}\dim D &= \left(\sum_{i=1}^{N} 4d_i + n_i + s_i \right) - 3s \\ &= 4d + n - s \\ &\leq 4d + n = \operatorname{vdim} \overline{M}_{1,n}(\mathbb{P}^3, d).\end{aligned}$$

   Equality holds only for $s = 0$, but this is not possible as the curve would be rational in that case. Thus we see that D is not of excessive dimension.

2. Now assume C is a tree and without loss of generality let C_1 be the elliptic

twig. There are two sub-cases here: first assume $d_1 > 0$. Then

$$\begin{aligned}\dim D &= 4d_1 + n_1 + s_1 + \left(\sum_{i=2}^{N} 4d_i + n_i + s_i\right) - 3s \\ &= 4d + n - s,\end{aligned}$$

which is the same result as above. Now however $s = 0$ is possible – in this case D is the locus of smooth curves.

Finally let $d_1 = 0$. Then

$$\begin{aligned}\dim D &= 3 + n_1 + s_1 + \left(\sum_{i=2}^{N} 4d_i + n_i + s_i\right) - 3s \\ &= 4d + n + 3 - s.\end{aligned}$$

We see that D is of excessive dimension if and only if $s \leq 3$. A priori there are seven types of curves satisfying these conditions, however some of the corresponding components are contained in the boundary of higher dimensional ones, see figure 2.2. This leaves us precisely with the components claimed. □

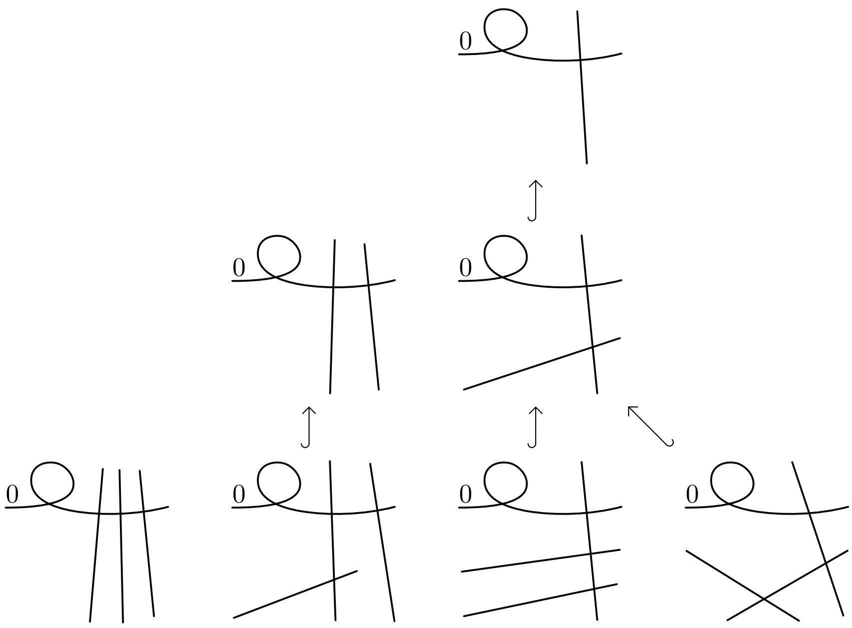
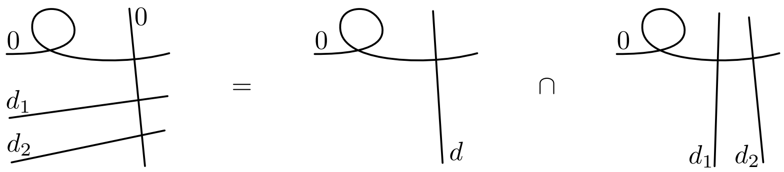


Figure 2.2: Inclusion structure of relevant components.

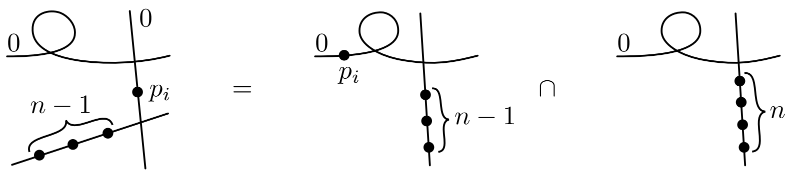
Remark 2.15. Any two components of types D_i and D_j are not contained in each other or any larger component, but they may intersect non-trivially. Consider the general element of a type D_2 component and let the nodes approach each other. In the limit a contracted rational component connecting all three twigs will fork off. This curve however is contained in the boundary of a type D_1 component, compare pictographic equation (2.6). The intersection is still of excessive dimension:

$$\dim(D_1 \cap D_2) = \operatorname{vdim} \overline{M}_{1,n}(X, d).$$



(2.6)

In the presence of marked points we additionally get an intersection between components of type D_1, see pictographic equation (2.7). Here, too, the intersection is of excessive dimension: $\operatorname{vdim} \overline{M}_{1,n}(X, d) + 1$.



(2.7)

Chapter 3

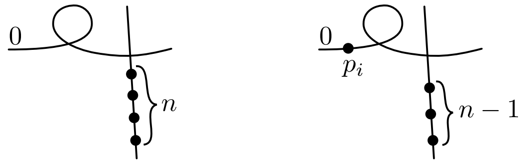
The Virtual Fundamental Class

In this chapter we will compute the contribution of each of the excessive dimensional components to the Gromov-Witen invariants. In the end, the main theorem 1.28 will be fully proven. We proceed as follows:

1. Show that components of type D_2 and D_3 do never contribute and D_1-type components only contribute if the number of marks on the elliptic twig is at most 1 (theorem 3.7).

2. Let D_R be the component (there is only one) of type D_1 whose general element has all marks placed on the rational twig, see figure 3.1. Then the contribution of D_R is determined in theorem 3.22.

3. Finally there are components $D_{E,i}$ of type D_1 characterized by a general element with all marks except p_i placed on the rational twig. In theorem 3.25 we will see that the combination of D_R with all of the $D_{E,i}$-components produces exactly the coefficient from the main theorem. The proof of the latter will follow easily.



(a) Component D_R. (b) Component $D_{E,i}$.

Figure 3.1: Notation for special components of type D_1. Now D_R and $D_{E,i}$ denote indeed components and not just types.

From now on our only goal is the main theorem, thus we will specialize the assumptions on the ambient variety even more by defining the following short hand notation:

$$\boxed{\begin{array}{c} X := \mathbb{P}^3 \text{ and if } n \text{ and } d \text{ are fixed but arbitrary} \\ M := M_{1,n}(X,d) \text{ and } \overline{M} := \overline{M}_{1,n}(X,d). \end{array}}$$

3.1 Reducing to $D_R \cup D_{E,1} \cup \cdots \cup D_{E,n}$

In corollary 2.14 we identified all of the components of excessive dimension. A priori, any of these components could contribute to the virtual fundamental class of $\overline{M}$ and thereby to the Gromov-Witten invariants. We will show in this section that for most of these components this is not the case.

Definition 3.1. Let $m \in \mathbb{N}_{\geq 0}$ and $\alpha,\ \beta \in A_m(\overline{M})$. We define an equivalence relation on $A_m(\overline{M})$ by

$$\alpha \bowtie \beta \quad :\Longleftrightarrow \quad \forall \gamma_1, \ldots, \gamma_n \in A^*(X): \int_\alpha ev_1^*\gamma_1 \cdots ev_n^*\gamma_n = \int_\beta ev_1^*\gamma_1 \cdots ev_n^*\gamma_n.$$

If $\alpha \bowtie \beta$ then we will say that they are **enumeratively equivalent**. A class $\alpha \in A_m(\overline{M})$ is said to be **enumeratively irrelevant** if $\alpha \bowtie 0$. A closed subscheme $S \subseteq \overline{M}$ is called **enumeratively irrelevant** if every $\alpha \in A_{\mathrm{vdim}\,\overline{M}}(S)$ is enumeratively irrelevant.

From the definition we can see immediately that subschemes of dimension less than vdim $\overline{M}$ are enumeratively irrelevant. Furthermore, every closed schubscheme $S' \subseteq S$ of an enumeratively irrelevant subscheme S is irrelevant itself as well.

Remark 3.2. Until now we used the term "virtual fundamental class" only for the entire moduli space $\overline{M}$. As we are interested in braking this construct down into pieces defined on irreducible components of $\overline{M}$ we need to refine this notation. For this recall that in chapter 2 we stated the long exact hyper-Ext sequence (2.4) which stems from an underlying exact sequence of sheaves (remark 2.5). In this sequence the terms $\mathcal{E}^1$ and $\mathcal{E}^2$ gave rise to the virtual dimension (compare definition 2.9). Furthermore, these terms govern the virtual fundamental class $[\overline{M}]^{virt}$ as well. In particular, for any $U \subseteq \overline{M}$ open we may restrict the sequence of sheaves to U and hence introduce a virtual fundamental class of $[U]^{virt}$. If $i : U \hookrightarrow \overline{M}$ is the inclusion, then the virtual fundamental classes are related by pull-back:

$$i^*[\overline{M}]^{virt} = [U]^{virt}.$$

The notion of enumerative irrelevance allows for an even closer connection between $[\overline{M}]^{virt}$ and $[U]^{virt}$. This an easy corollary from the following well-known lemma ([Gat03a, lemma 9.1.13]):

Lemma 3.3. *Let X be a scheme, $Y \subseteq X$ a closed subscheme. Then for all $k \geq 0$ there is an exact sequence*

$$A_k(Y) \to A_k(X) \to A_k(X \setminus Y) \to 0.$$

Corollary 3.4. *Let $S \subseteq \overline{M}$ be an enumeratively irrelevant, closed subscheme and denote $U := \overline{M} \setminus S$. Then*

$$[\overline{M}]^{virt} \bowtie [U]^{virt}.$$

Proof. Use lemma 3.3 with $Y = S$ and $X = \overline{M}$. Let $i : S \hookrightarrow \overline{M}$ be the inclusion. Note that the image of $[\overline{M}]^{virt}$ in U is by construction the restriction of the virtual fundamental class to U, which is just $[U]^{virt}$. By exactness there exists a (not necessarily unique) $\alpha \in A_{\operatorname{vdim} \overline{M}}(S)$ such that

$$[\overline{M}]^{virt} = [U]^{virt} + \alpha.$$

As S was assumed to be enumeratively irrelevant, the claim follows. □

Let us now proceed to the theorem claimed by the title of this section. The key ingredient for the proof is a trick based on the projection formula from intersection theory [Har77, theorem A 1.1.]:

Lemma 3.5 (projection formula). *Let $\varphi : M \to Y$ be a proper morphism of non-singular, quasi-projective varieties M and Y. Then*

$$\forall x \in A_*(M), y \in A^*(Y): \qquad \varphi_*(x \cdot \varphi^* y) = \varphi_* x \cdot y.$$

The following lemma uses the projection formula to establish a useful tool to prove enumerative insignificance of a given closed subscheme $D \subseteq \overline{M}$. The idea is to shift the evaluation of pulled-back conditions from D to some space where it has to be trivial for dimensional reasons.

Lemma 3.6. *Let D be a closed subscheme of $\overline{M}$. Assume there exists a smooth, projective variety Y together with morphisms*

$$\begin{aligned} \varphi &: D \to Y \\ \widetilde{ev}_i &: Y \to X, \quad i = 1, \dots, n \end{aligned}$$

such that all of the diagrams

$$\begin{array}{ccc} D & \xrightarrow{\varphi} & Y \\ & {\scriptstyle ev_i}\searrow & \downarrow{\scriptstyle \widetilde{ev}_i} \\ & & X \end{array}$$

commute. If $\dim Y < \operatorname{vdim} \overline{M}$ then D is enumeratively insignificant.

Proof. Because of the evaluation maps factoring through φ we may write

$$ev_1^*\gamma_1 \cdot \dots \cdot ev_n^*\gamma_n = \varphi^*\big(\underbrace{\widetilde{ev}_1^*\gamma_1 \cdot \dots \cdot \widetilde{ev}_n^*\gamma_n}_{=:T}\big).$$

Now use the projection formula with $\alpha \in A_{\operatorname{vdim} \overline{M}}(D)$ arbitrary:

$$\varphi_*(\underbrace{\varphi^* T \cdot \alpha}_{\text{dimension } 0}) = T \cdot \varphi_* \alpha. \tag{3.1}$$

As the dimension of Y is strictly smaller than the virtual dimension of the moduli space, $\varphi_*\alpha \in A_{\operatorname{vdim} \overline{M}}(Y) = 0$ is necessarily trivial and with it the entire right hand side of equation (3.1). But the push-forward of 0-dimensional cycles preserves degree (points are always mapped to points), hence

$$\deg(\varphi^* T \cdot \alpha) = \deg \varphi_*(\varphi^* T \cdot \alpha) = \deg(T \cdot \varphi_* \alpha) = 0.$$

The γ_i were arbitrary, thus the claim follows. □

With lemma 3.6 at hand we only need to find a suitable space Y and morphism φ in order to exclude the possibility of an irreducible component D contributing to the Gromov-Witten invariants. This is done in the proof of the next theorem for all excessive dimensional components at once. The idea behind the construction is to remove the contracted elliptic twig from the general element. The fact that the twig is contracted means that it does not really contribute to enumerative questions, even though it does contribute to the excessive dimension of D.

Theorem 3.7. *Let $D \subseteq \overline{M}$ be a component of excessive dimension. Unless it is of type D_1 with its general element having at most one mark on the elliptic twig, D is enumeratively insignificant.*

Proof. Let $f : C \to X$ be a general element of D. By corollary 2.14 we know that C has $s \leq 3$ nodes and is a tree with the elliptic twig being contracted. Let n_E denote the number of marks on the elliptic twig. The component D can be expressed as a fibred product with respect to evaluation maps corresponding to successive attachment of rational twigs to the elliptic one:

$$\begin{aligned} D &\cong \Big(\cdots \big(cl(M_{1,n_E+s}(X,0)) \times_X T_1 \big) \times_X \cdots \Big) \times_X T_s \\ &\cong cl(M_{1,n_E+s}(X,0)) \times_X \big(\underbrace{T_1 \times_X \cdots \times_X T_s}_{=:Y} \big), \end{aligned}$$

where the T_i are of the form $\overline{M}_{0,n_i+1}(X, d_i)$ for suitable n_i and d_i. Note that Y may not be a subset of a moduli space of stable maps anymore as marks may coincide or more than two twigs may be glued in a single point. Nevertheless we may still consider elements of T as maps $f' : (C', \underline{p}') \to X$ defined on an n-pointed curve C'.

We define evaluation maps $\widetilde{ev}_i : Y \to X$ which are given by $\widetilde{ev}_i(f') := f'(p'_i)$. With this the evaluation maps ev_i on D factor through the projection

$$\varphi : D \to Y$$

obtained from the representation (15), i.e. $ev_i = \widetilde{ev}_i \circ \varphi$. Let us compute the dimension of Y using the techniques from chapter 2:

$$\begin{aligned} \dim D &= \dim M_{1,n_E+s}(X,0) + \dim Y - 3 \\ \iff \dim Y &= \dim D - \dim M_{1,n_E+s}(X,0) + 3 \\ &= (4d + n + 3 - s) - (3 + n_E + s) + 3 \\ &= 4d + (n - n_E) + (3 - 2s). \end{aligned}$$

If this number is compared to $\operatorname{vdim} \overline{M}_{1,n}(X,d)$ we obtain exactly the assumption of the claim:

$$\dim Y < \operatorname{vdim} \overline{M}_{1,n}(X,d) \iff (s = 1 \wedge n_E > 2) \vee s > 1.$$

We may now apply lemma 3.6 to complete the proof. □

Remark 3.8. By theorem 3.7 the only relevant components are M, D_R, and D_{E_i}. Let S denote the union of all other irreducible components of $\overline{M}$ together with the locus of intersection between the closure $cl(M)$ of M and $D_R \cup D_{E,1} \cup \cdots \cup D_{E,n}$. Now S is enumeratively irrelevant (partly due to dimensional reasons) and thus we get by corollary 3.4

$$\begin{aligned}[\overline{M}]^{virt} &\bowtie [\overline{M} \setminus S]^{virt} \\ &= \left[M \,\dot{\cup}\, \big((D_R \cup D_{E,1} \cup \cdots \cup D_{E,n}) \setminus S\big)\right]^{virt} \\ &= [M]^{virt} + \left[\big((D_R \cup D_{E,1} \cup \cdots \cup D_{E,n}) \setminus S\big)\right]^{virt}.\end{aligned}$$

Again by corollary 3.4 we may now pass under enumerative equivalence to the closure. The resulting classes are technically not virtual fundamental classes but we will still denote them with the same symbol, in particular we use the notation

$$[D_R \cup D_{E,1} \cup \cdots \cup D_{E,n}]^{virt} \in A_{\operatorname{vdim} \overline{M}}(D_R \cup D_{E,1} \cup \cdots \cup D_{E,n})$$

to denote a class that is equivalent to the virtual fundamental class. With this notation we may write

$$[\overline{M}]^{virt} \bowtie [M]^{virt} + [D_R \cup D_{E,1} \cup \cdots \cup D_{E,n}]^{virt}. \tag{3.2}$$

Remark 3.9. The map constructed in the proof of theorem 3.7 is defined on the components D_R and $D_{E,i}$ as well. In addition to the construction in the proof we may forget the point of attachment in case of D_R such that we have in either case a map to $\overline{M}_{0,n}(X,d)$. This map will be important in the following sections. We refer to it as **contraction of** E and denote it by φ.

3.2 The Contribution of D_R

The general construction of the virtual fundamental class is carried out in [BF97] and relies heavily on the formalism of stacks which we cannot cover in this thesis. However [BF97, proposition 5.6] is a special case that is of interest to us:

Proposition 3.10. *Let M be a smooth Deligne-Mumford stack with a perfect obstruction theory $E^\bullet$. Then $h^1(E^{\bullet\vee})$ is locally free of rank, say, r and*

$$[M]^{virt} = c_r\big(h^1(E^{\bullet\vee})\big) \cdot [M].$$

In our case D_R may be rewritten as product of smooth spaces

$$D_R = \overline{M}_{1,1} \times \overline{M}_{0,1+n}(X,d), \tag{3.3}$$

in particular D_R itself is smooth as well. Let $\mathcal{E}^i$ denote the sheaf with fibre E^i given the hyper-Ext sequence (2.4). Application of proposition 3.10 to D_R gives the following formula

$$[D_R]^{virt} = c_2(\mathcal{E}^2|_{D_R}).$$

Remark 3.11. Another case were proposition 3.10 can be applied is that of (the topological closure of) $M_{1,n}(X,d)$. By theorem 2.12 we know that $\mathcal{E}^2$ is trivial over the smooth locus, i.e. a vector bundle of rank 0. Thus

$$[M_{1,n}(X,d)]^{virt} = [M_{1,n}(X,d)].$$

Notation 3.12. Throughout this section we will work on D_R and use a general element $f : C \to X$ in computations. By definition of D_R the curve C is a union $E \cup R$ of a smooth elliptic twig E and a smooth rational twig R. Let $\nu \in E \cap R$ denote the node. We will keep this notation throughout this section.

Lemma 3.13. *For any line bundle $\mathcal{L}$ on C there is a short exact sequence of O_C-modules*

$$0 \to \mathcal{L} \to \mathcal{L}|_E \oplus \mathcal{L}|_R \to \mathcal{L}_\nu \to 0.$$

Lemma 3.14. $H^1(C, \mathcal{O}_C) \cong H^1(E, \mathcal{O}_E)$.

Proof. If we take $\mathcal{L}$ in lemma (3.13) to be $\mathcal{O}_C$ we get

$$0 \to \mathcal{O}_C \to \mathcal{O}_E \oplus \mathcal{O}_R \to \mathcal{O}_{C,\nu} \to 0$$

which induces

$$\cdots \to H^1(C, \mathcal{O}_C) \to H^1(E, \mathcal{O}_E) \oplus H^1(R, \mathcal{O}_R) \to 0. \tag{3.4}$$

Here one should note [Liu02, exercise 5.2.3] to formally justify switching from C to E and R inside the H^i. Recall that for any curve S by definition $g(S) = \dim H^1(S, \mathcal{O}_S)$. Hence (3.4) is a surjective morphism from a one-dimensional vector space to another and hence an isomorphism. □

Lemma 3.15. *Let $p := f(\nu) \in X$. Then*

$$H^1\big(C, f^*\mathcal{O}_X(1)\big) \cong H^1(E, \mathcal{O}_E) \otimes_k T_{X,p}.$$

Proof. Again we want to use lemma (3.13), this time with $\mathcal{L} = f^*\mathcal{O}_X(1)$. We start by computing the occurring terms.

Note that $(f^*\mathcal{O}_X(1))|_E = (f|_E)^*\mathcal{O}_X(1)$ and the same for the restriction to R. The former maps with degree 0, the latter with degree d. For the elliptic twig we get

$$\begin{aligned}(f|_E)^*\mathcal{O}_X(1) &\overset{\text{def}}{=} \mathcal{O}_E \otimes_{(f|_E)^{-1}\mathcal{O}_X} (f|_E)^{-1}\mathcal{O}_X(1)\\ &= \mathcal{O}_E \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_X(1)_p\\ &= \mathcal{O}_E \otimes_k \mathcal{O}_X(1)_p,\end{aligned}$$

while the rational twig yields

$$(f|_R)^*\mathcal{O}_X(1) = \begin{cases} \mathcal{O}_R\big((f|_R)^*H\big) = \mathcal{O}_R(d) & \text{if } d > 0\\ \mathcal{O}_R \otimes_k \mathcal{O}_X(1)_p & \text{if } d = 0, \text{ analogously to } E \text{ above}\end{cases}$$

In either case $H^1(R, (f|_R)^*\mathcal{O}_X(1)) = 0$. Thus we may assume for simplicity $d > 0$ as it does not change the result.

Now the sequence from lemma (3.13) becomes

$$0 \to f^*\mathcal{O}_X(1) \to \big(\mathcal{O}_E \otimes_k \mathcal{O}_X(1)_p\big) \oplus \mathcal{O}_R(d) \to \mathcal{O}_X(1)_p \to 0.$$

Again we get an induced sequence in cohomology (numbers below terms denote dimension)

$$
\begin{aligned}
0 \to H^0\big(C, f^*\mathcal{O}_X(1)\big) \to &\Big(\underbrace{H^0(E,\mathcal{O}_E)\otimes \mathcal{O}_X(1)_p}_{1}\Big) \oplus \underbrace{H^0\big(R,\mathcal{O}_R(d)\big)}_{d+1} \\
&\stackrel{(*)}{\to} \underbrace{H^0\big(C,\mathcal{O}_X(1)_p\big)}_{1} \to H^1\big(C, f^*\mathcal{O}_X(1)\big) \\
&\stackrel{(\Diamond)}{\to} \Big(\underbrace{H^1(E,\mathcal{O}_E)}_{1}\otimes_k \underbrace{\mathcal{O}_X(1)_p}_{1}\Big) \oplus \underbrace{H^1\big(R,\mathcal{O}_R(d)\big)}_{0} \to 0, \qquad (3.5)
\end{aligned}
$$

where we used that $\mathcal{O}_X(1)_p$ is a finite dimensional k-vector space and hence can be pulled out of the cohomology ([Liu02, lemma 5.2.26]). For the dimensions use Riemann-Roch and for $R \cong \mathbb{P}^1$, [Liu02, lemma 5.3.1] tells us

$$H^0\big(R,\mathcal{O}_R(d)\big) \cong k[x,y]_d.$$

The ring of global sections of the skyscraper sheaf is one-dimensional because it is a line bundle supported in only one point, hence just a one-dimensional vector space.

By exactness of (3.5) we see that $(\Diamond)$ is surjective. We claim that it is an isomorphism. In order to prove this we check that the map $(*)$ is surjective. It is given by

$$
\begin{aligned}
\Big(\Gamma(\mathcal{O}_E)\otimes_k \mathcal{O}_X(1)_p\Big) \oplus \Gamma\big(\mathcal{O}_R(d)\big) &\to \Gamma\big(\mathcal{O}_X(1)_p\big) \cong \mathcal{O}_X(1)_p \\
e\otimes c + r &\mapsto ec - r(\nu).
\end{aligned}
$$

Now note that $\Gamma(\mathcal{O}_E) \cong \mathcal{O}_X(1)_p \cong k$ and hence every element of $\mathcal{O}_X(1)_p$ has a pre-image even in the first direct summand. □

Lemma 3.16. *Let C be a reduced, nodal curve consisting of two twigs C_1 and C_2 meeting transversally in a single node ν. Then*

$$T_C \cong T_{C_1}(-\nu) \oplus T_{C_2}(-\nu).$$

Proof. By [Har10, lemma 27.6] we have

$$T_C \cong (\mathcal{I}_{\nu,C_1}\otimes T_{C_1}) \oplus (\mathcal{I}_{\nu,C_2}\otimes T_{C_2}),$$

where $\mathcal{I}_{\nu,C_i}$ denotes the ideal sheaf of ν in C_i. An elementary computation in the local setting $C_i \cong \operatorname{Spec} k[x]$ shows that $\mathcal{I}_{\nu,C_i} = \mathcal{O}_{C_i}(-\nu)$. □

The next theorem expresses the sheaf $\mathcal{E}^2$ with respect to the vector bundles defined in section 1.4. Note that all occurring classes live a priori on one of the factors of D_R in equation (3.3). In order to get classes on D_R we implicitly use the pull-back along the projections.

Notation 3.17. Recall and define notation:

- let $\mathbb{E}$ be the Hodge bundle on $\overline{M}_{1,1}$ with $c_1(\mathbb{E}) = \lambda$,

- let $\mathbb{T}_E$ be the bundle on $\overline{M}_{1,1}$ defining the (only) psi-class, i.e. $c_1(\mathbb{T}_E) = \psi \in A^1(\overline{M}_{1,1})$,
- $\mathbb{T}_R$ analogously the bundle on $\overline{M}_{0,n+1}(X,d)$ giving rise to the $(n+1)$-st psi-class, for convenience denoted by ψ_1,
- and let $H \subseteq X$ denote a hyperplane and $ev : \overline{M}_{0,n+1}(X,d) \to X$ be the evaluation map corresponding to the $(n+1)$-st mark, such that $ev^*H \in A^1(\overline{M}_{0,n+1}(X,d))$.

Theorem 3.18. *The sheaf $\mathcal{E}^2$ is given by*

$$\mathcal{E}^2 \cong \Big(\big(\mathbb{E}^\vee \otimes ev^*H\big)^{\oplus 4}/\beta(\mathbb{E}^\vee)\Big)/\alpha\big(\mathbb{T}_E^\vee \otimes \mathbb{T}_R^\vee\big),$$

where the maps α and β are given for each fibre in sequences (3.6) *and* (3.7) *below.*

Proof. We proof the expression from the claim for every fibre using natural identifications only. By naturality the result will lift to an isomorphism of vector bundles.

Start by looking at the final three terms of the long exact hyper-Ext sequence (2.4):

$$\cdots \to \operatorname{Ext}^1\big(\Omega_C(P), \mathcal{O}_C\big) \xrightarrow{\alpha} H^1(C, f^*T_X) \to E^2 \to 0. \tag{3.6}$$

Next pull back the Euler sequence on X along f to obtain

$$0 \to \mathcal{O}_C \to f^*\mathcal{O}_X(1)^{\oplus 4} \to f^*T_X \to 0,$$

which then induces

$$\cdots \to H^1(C, \mathcal{O}_C) \xrightarrow{\beta} H^1\big(C, f^*\mathcal{O}_X(1)\big)^{\oplus 4} \to H^1(C, f^*T_X) \to 0. \tag{3.7}$$

In both sequence (3.6) and (3.7) we may interpret exactness as the final term being the quotient of the previous ones. Combining them we obtain a first expression for E^2:

$$E^2 \cong \Big(\Big(H^1\big(C, f^*\mathcal{O}_X(1)\big)\Big)^{\oplus 4}/\beta\big(H^1(C,\mathcal{O}_C)\big)\Big)/\alpha\Big(\operatorname{Ext}^1\big(\Omega_C(P), \mathcal{O}_C\big)\Big).$$

By lemma 3.14 and lemma 3.15 this can be refined to

$$E^2 \cong \Big(\Big(H^1(E, \mathcal{O}_E) \otimes_k \mathcal{O}_X(1)_p\Big)^{\oplus 4}/\beta\big(H^1(E,\mathcal{O}_E)\big)\Big)/\alpha\Big(\operatorname{Ext}^1\big(\Omega_C(P), \mathcal{O}_C\big)\Big) \tag{3.8}$$

such that all that is left to show is

$$\alpha\Big(\operatorname{Ext}^1\big(\Omega_C(P), \mathcal{O}_C\big)\Big) \cong \alpha\big(T_{E,\nu} \otimes T_{R,\nu}\big).$$

We resolve the global Ext with help of the local-to-global Ext spectral sequence from proposition B.4. Remark B.5 gives a decomposition

$$\operatorname{Ext}^1\big(\Omega_C(P), \mathcal{O}_C\big) \cong \Gamma\big(\underbrace{\mathcal{E}xt^1(\Omega_C(P), \mathcal{O}_C)}_{\text{skyscraper sheaf}}\big) \oplus H^1\big(C, \underbrace{\mathcal{H}om(\Omega_C(P), \mathcal{O}_C)}_{=T_C(-P)}\big). \tag{3.9}$$

The skyscraper sheaf is given by [HM98, proposition 3.31]

$$\Gamma\big(\mathcal{E}xt^1(\Omega_C(D), \mathcal{O}_C)\big) \cong T_{E,\nu} \otimes T_{R,\nu}.$$

We claim that the cohomology with respect to $T_C(-P)$ lies in the kernel of α. By lemma 3.16 it decomposes as

$$H^1\big(C, T_C(-P)\big) = H^1\big(E, T_E(-P-\nu)\big) \oplus H^1\big(R, T_R(-P-\nu)\big). \tag{3.10}$$

But now E and R are smooth and thus we know by theorem 2.12 that their deformations lift to deformations of the stable maps $f|_E$ and $f|_R$ respectively. This completes the proof. □

Remark 3.19. We would like to point out, that the decompositions in equations (3.9) and (3.10) come with geometric interpretation:

- $\mathrm{Ext}^1(\Omega_C(P), \mathcal{O}_C))$ is the space of first-order deformations of the pointed curve C,
- $T_{E,\nu} \otimes T_{R,\nu}$ are resolutions of the singularity,
- $H^1(E, T_E(-P-\nu))$ are deformations of E with an additional mark placed in ν and
- $H^1(R, T_R(-P-\nu))$ analogously for R.

This means that a deformation of C consists of deformations of the twigs together with a resolution of the singularity.

We have achieved a natural expression for $\mathcal{E}^2$ and we may now proceed to compute its second Chern class. For this we use basic computation rules for Chern clases, quickly summarized in the following lemma (statements from [Har77, section A.3]).

Lemma 3.20. *Let $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be vector bundles on some non-singular, quasi-projective variety and let $(\alpha_i)_i$ and $(\beta_j)_j$ denote formal elements such that*

$$c(\mathcal{E}) = \prod_{i=1}^{\operatorname{rank}\mathcal{E}} (1+\alpha_i) \qquad \textit{and} \qquad c(\mathcal{F}) = \prod_{i=1}^{\operatorname{rank}\mathcal{F}} (1+\beta_i).$$

*These representations exist by the splitting principle. For reference we call the α_i and β_i **Chern roots** of $\mathcal{E}$ and $\mathcal{F}$ respectively (compare [Gat03a, remark 10.3.9]).*

a) Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be exact. Then $c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G})$. Note that the maps are of no importance.

b) In particular $c(\mathcal{E} \oplus \mathcal{F}) = c(\mathcal{E})c(\mathcal{F})$.

c) The Chern roots of $\mathcal{E} \otimes \mathcal{F}$ are given by $(\alpha_i\beta_j)_{i,j}$.

d) The Chern roots of $\mathcal{E}^\vee$ are $(-\alpha_i)_i$.

e) The Chern roots of a pull-back of a vector bundle are the pull-backs of the Chern roots.

Theorem 3.21. *With notation 3.17 we have*

$$c_2(\mathcal{E}^2) = -8H\lambda + 4H\psi + 4H\psi_1 + 3\lambda^2 - 3\lambda\psi - 3\lambda\psi_1 + 6H^2 + \psi^2 + 2\psi\psi_1 + \psi_1^2.$$

Proof. For each of the building blocks of the expression derived in theorem 3.18 we know the total Chern class (compare section 1.4):

$$\begin{aligned} c(\mathbb{E}) &= 1 + \lambda, \\ c(\mathbb{T}_E) &= 1 + \psi, \\ c(\mathbb{T}_R) &= 1 + \psi_1, \\ c\big(\mathcal{O}_X(1)\big) &= 1 + H. \end{aligned}$$

Using lemma 3.20 part d) we obtain

$$\begin{aligned} c(\mathbb{E}^\vee) &= 1 - \lambda, \\ c(\mathbb{T}_E^\vee) &= 1 - \psi \text{ and} \\ c(\mathbb{T}_R^\vee) &= 1 - \psi_1, \end{aligned} \tag{3.11}$$

and by lemma 3.20 part e)

$$c\big(ev^*\mathcal{O}_X(1)\big) = 1 + ev^*H. \tag{3.12}$$

To enhance readability we abuse notation and denote ev_1^*H by H again. Now we apply the Chern calculus to the expression derived in theorem 3.18.

$$\begin{aligned} c(\mathcal{E}^2) &= \frac{c\big((\mathbb{E}^\vee \otimes H)^{\oplus 4}\big)}{c(\mathbb{E}^\vee)c(\mathbb{T}_E^\vee \otimes \mathbb{T}_R^\vee)} && \text{by lemma 3.20 part a)} \\ &= \frac{c(\mathbb{E}^\vee \otimes H)^4}{c(\mathbb{E}^\vee)c(\mathbb{T}_E^\vee \otimes \mathbb{T}_R^\vee)} && \text{by lemma 3.20 part b).} \end{aligned} \tag{3.13}$$

Finally we combine the expressions (3.11), (3.12), and (3.13) and use lemma 3.20 part c) to get

$$c(\mathcal{E}^2) = \frac{(1 - \lambda + H)^4}{(1 - \lambda)(1 - \psi - \psi_1)}. \tag{3.14}$$

Elements from the denominator can be inverted using the geometric series

$$\frac{1}{1-\theta} = \sum_{i=0}^{\infty} \theta^i = 1 + \theta + \dots + \theta^{\dim D}$$

for any $\theta \in A^1(D)$. Thus expression (3.14) can be expanded. However, as we are ultimately only interested in the second Chern class of E^2 we only need to find codimension 2 terms. The classes λ, ev_1^*H, ψ and ψ_1 are all of codimension 1, hence it suffices to find the quadratic terms in these four symbols.

$$\begin{aligned} c(E^2) &= \big(1 + (H - \lambda)\big)^4 \big(1 + \lambda + \lambda^2 + \cdots\big) \\ &\quad \cdot \big(1 + (\psi + \psi_1) + (\psi + \psi_1)^2 + \cdots\big) \\ &= \big(1 + 4(H - \lambda) + 6(H - \lambda)^2 + \cdots\big) \\ &\quad \cdot \big(1 + (\lambda + \psi + \psi_1) + \lambda^2 + (\lambda + \psi + \psi_1)(\psi + \psi_1) + \cdots\big) \\ &= \cdots + 4(H - \lambda)(\lambda + \psi + \psi_1) + 6(H - \lambda)^2 \\ &\quad + \lambda^2 + (\lambda + \psi + \psi_1)(\psi + \psi_1) + \cdots \end{aligned}$$

From here the claim can be read off. □

The next theorem completes the task of determining the contribution of D_R to Gromov-Witten invariants. Note that in case $n = 0$ it yields precisely the $N_{ab}^{(0)}$-coefficient from the main theorem 1.28.

Theorem 3.22. *Let $\varphi : D_R \to \overline{M}_{0,n}(X, d)$ be the contraction of E from remark 3.9. Then*

$$\varphi_* c_2(\mathcal{E}^2) = \frac{1}{24}(-4d + 2 - n)[\overline{M}_{0,n}(X, d)].$$

Proof. The map φ forgets the j-invariant of E as well as the point of attachment of E to R, i.e. the dimensions are

$$D_R = \underbrace{\overline{M}_{1,1}}_{1} \times \underbrace{\overline{M}_{0,n+1}(X, d)}_{4d+n+1} \to \underbrace{\overline{M}_{0,n}(X, d)}_{4d+n}.$$

More precisely φ loses one dimension in either factor of D_R. Thus a codimension 2 cycle in D_R yielding something non-trivial in $A^0(\overline{M}_{0,n}(X, d))$ has to be of codimension 1 in both $\overline{M}_{1,1}$ and $\overline{M}_{0,n+1}(X, d)$. For the expression from theorem 3.21 this means that only $H\lambda$, $H\psi$, $\psi_1\lambda$ and $\psi_1\psi$ contribute to $\varphi_* c_2(\mathcal{E}^2)$:

$$\varphi_* c_2(\mathcal{E}^2) = -8\varphi_*(H\lambda) + 4\varphi_*(H\psi) - 3\varphi_*(\lambda\psi_1) + 2\varphi_*(\psi\psi_1). \tag{3.15}$$

Further note that φ factors as

$$\begin{array}{ccc} D_R = \overline{M}_{1,1} \times \overline{M}_{0,n+1}(X, d) & \xrightarrow{\quad\varphi\quad} & \overline{M}_{0,n}(X, d) \\ & \searrow^{\tau} \quad \nearrow_{\pi} & \\ & \overline{M}_{0,n+1}(X, d). & \end{array}$$

Now recall that $\overline{M}_{1,1} \cong \mathbb{P}^1$, i.e. λ and ψ are 0 dimensional. This means that push forwards of intersection products from expression (3.15) along τ are just $\deg \lambda$ and $\deg \psi$ times H and ψ_1 respectively. From lemma 1.31 and corollary 1.35 these degrees are known to be

$$\deg \lambda = \deg \psi = \frac{1}{24}.$$

The push-forwards of H and ψ_1 along π are known as well (lemma 1.37 and lemma 1.32):

$$\begin{aligned} \pi_* H &= d[\overline{M}_{0,n}(X, d)] \\ \pi_* \psi_1 &= (-2 + n)[\overline{M}_{0,n}(X, d)]. \end{aligned}$$

All in all we have

$$\begin{aligned} \varphi_* c_2(\mathcal{E}^2) &= \left(-\frac{8}{24}d + \frac{4}{24}d - \frac{3}{24}(-2 + n) + \frac{2}{24}(-2 + n)\right)[\overline{M}_{0,n}(X, d)] \\ &= \frac{1}{24}(-4d + 2 - n)[\overline{M}_{0,n}(X, d)]. \end{aligned}$$

□

3.3 Proof of the Main Theorem

In order to proceed to a proof of the main theorem 1.28 we need to determine the influence of the components D_R and $D_{E,i}$ combined. In case $n = 0$ this is of course trivial – D_R is the only component of type D_1. The following property of virtual fundamental classes is a special case of [Beh97, axiom IV] and enables us to approach the task by induction.

Proposition 3.23. *Let $\pi : \overline{M}_{g,n+1}(X,d) \to \overline{M}_{g,n}(X,d)$ a forgetful map. Then*

$$\begin{array}{ccc} \overline{M}_{g,n+1}(X,d) & \xrightarrow{\pi} & \overline{M}_{g,n}(X,d) \\ \downarrow & \square & \downarrow \\ \overline{M}_{g,n+1} & \longrightarrow & \overline{M}_{g,n} \end{array}$$

is a cartesian diagram and it holds

$$[\overline{M}_{g,n+1}(X,d)]^{virt} = \pi^*[\overline{M}_{g,n}(X,d)]^{virt}.$$

Remark 3.24. The reader might be worried that the $\overline{M}_{g,n}$ used in the above diagram does not exist due to lack of marks. This may be true in the scheme sense, however it does still exist as a stack and the statement remains true.

Intuitively proposition 3.23 tells us that for any $(C, \underline{p}, f)$ contained in the support of a cycle representing $[\overline{M}_{g,n-1}(X,d)]^{virt}$ any stable map arising from C by adding a new mark at any position will be contained in a cycle representation of $[\overline{M}_{g,n}(X,d)]^{virt}$.

The next theorem solves the remaining problem.

Theorem 3.25. *Let $\varphi : D_R \cup D_{E,1} \cup \cdots \cup D_{E,n} \to \overline{M}_{0,n}(X,d)$ denote the contraction of E from remark 3.9. Then*

$$\varphi_*[D_R \cup D_{E,1} \cup \cdots \cup D_{E,n}]^{virt} = \frac{1-2d}{12}[\overline{M}_{0,n}(X,d)].$$

Proof. For $n = 0$ the claim follows from theorem 3.22. Hence assume $n > 0$.

Let T denote the union of all excessive dimensional, enumeratively not irrelevant component of $\overline{M}_{1,n-1}(X,d)$. Then the diagram from proposition 3.23 restricts to

$$\begin{array}{ccc} D_R \cup D_{E,1} \cup \cdots \cup D_{E,n} & \xrightarrow{\pi} & T \\ \downarrow & & \downarrow \\ \overline{M}_{g,n+1} & \longrightarrow & \overline{M}_{g,n}, \end{array}$$

where π is the morphism forgetting the n-th mark. Proposition 3.23 tell us that $\pi^*[T]^{virt} = [D_R \cup D_{E,1} \cup \cdots \cup D_{E,n}]^{virt}$. Note that by construction of pull-backs in intersection theory we always pull-back elements of A^*. This means that we tacitly used Serre duality, i.e. an intersection with the usual fundamental class. The latter however is by definition a sum over all irreducible components and thus we get

$$\begin{aligned} [D_R \cup D_{E,1} \cup \cdots \cup D_{E,n}]^{virt} &= \pi^*[T]^{virt} \cdot [D_R \cup D_{E,1} \cup \cdots \cup D_{E,n}] \\ &= \pi^*[T]^{virt} \cdot \big([D_R] + [D_{E,1}] + \cdots + [D_{E,n}]\big) \\ &= (\pi|_{D_R})^*[T]^{virt} + \cdots + (\pi|_{D_{E,n}})^*[T]^{virt}. \end{aligned} \tag{3.16}$$

Over D_R the follwoing diagram is cartesian:

$$\begin{array}{ccc} D_R & \xrightarrow{\pi|_{D_R}} & D \\ {\scriptstyle\varphi}\downarrow & \square & \downarrow \\ \overline{M}_{0,n+1}(X,d) & \longrightarrow & \overline{M}_{0,n}(X,d). \end{array}$$

Therefore we know that $\varphi_*\pi^*[T]^{virt} = \pi^*\varphi_*[T]^{virt}$, with the right hand side being given by induction.

Now we investigate $D_{E,i}$. For this note that $\varphi|_{D_{E,i}}$ factors through $\pi : D_{E,i} \to T$:

$$\begin{array}{ccc} D_{E,i} & \xrightarrow{\pi} & T \\ \downarrow{\scriptstyle\varphi} & \swarrow{\scriptstyle\tau} & \\ \overline{M}_{0,n}(X,d). & & \end{array}$$

This however means that the push-forward

$$\varphi_*(\pi|_{D_{E,i}})^*[T]^{virt} = \tau_* \underbrace{(\pi|_{D_{E,i}})_*(\pi|_{D_{E,i}})^*[T]^{virt}}_{=0} = 0$$

is trivial. Now apply φ_* to equation (3.16) to complete the proof. $\square$

This was the last ingredient we needed. We will now combine it with equation (3.2) to finally obtain a proof for our main theorem.

Proof (of theorem 1.28). Let $\gamma_1, \ldots, \gamma_n \in A^*(X)$ be classes of a points and b lines such that $4d = 2a + b$. Furthermore denote

$$D := D_R \cup D_{E,1} \cup \cdots \cup D_{E,n}$$

and let $\varphi : D \to \overline{M}_{0,n}(X,d)$ be the contraction of E. Then by equation (3.2) and remark 3.11

$$\begin{aligned} N_{ab}^{(1)} &= \int_{[\overline{M}_{1,n}(X,d)]^{virt}} ev_1^*\gamma_1 \cdots ev_n^*\gamma_n \\ &= \int_{[M_{1,n}(X,d)]+[D]^{virt}} ev_1^*\gamma_1 \cdots ev_n^*\gamma_n \\ &= \underbrace{\int_{[M_{1,n}(X,d)]} ev_1^*\gamma_1 \cdots ev_n^*\gamma_n}_{\text{number of elliptic curves satisfying the conditions } \gamma_i} + \int_{[D]^{virt}} ev_1^*\gamma_1 \cdots ev_n^*\gamma_n. \end{aligned}$$

For the remaining term on the right hand side we get

$$
\begin{aligned}
&\int_{[D]^{virt}} ev_1^*\gamma_1 \cdots ev_n^*\gamma_n \\
=&\deg \varphi_*\Big(\big(ev_1^*\gamma_1 \cdots ev_n^*\gamma_n\big)\cdot [D]^{virt}\Big) && \text{definition 1.25} \\
=&\deg \varphi_*\Big(\varphi^*\big(\widetilde{ev}_1^*\gamma_1 \cdots \widetilde{ev}_n^*\gamma_n\big)\cdot [D]^{virt}\Big) && \text{construction of } \varphi \\
=&\deg \Big(\big(\widetilde{ev}_1^*\gamma_1 \cdots \widetilde{ev}_n^*\gamma_n\big)\cdot \varphi_*[D]^{virt}\Big) && \text{projection formula} \\
=&\int_{\varphi_*[D]^{virt}} \widetilde{ev}_1^*\gamma_1 \cdots \widetilde{ev}_n^*\gamma_n && \text{definition 1.25} \\
=&\int_{\frac{1-2d}{12}[\overline{M}_{0,n}(X,d)]} \widetilde{ev}_1^*\gamma_1 \cdots \widetilde{ev}_n^*\gamma_n && \text{theorem 3.25} \\
=&\frac{1-2d}{12}N_{ab}^{(0)} && \text{definition 1.25.}
\end{aligned}
$$

Combining the computations proves the claim. □

Appendix A

The Ext Functor

Let R be a ring and M an R-module. Consider the Hom-functor

$$\operatorname{Hom}_R(-, M) : \mathfrak{Mod}(R) \to \mathfrak{Mod}(R)$$

and note that it is a contravariant, left-exact functor.

Definition A.1. The right derived functors of $\operatorname{Hom}_R(-, M)$ are denoted

$$\operatorname{Ext}^i_R(-, M) : \mathfrak{Mod}(R) \to \mathfrak{Mod}(R), \quad i \geq 0.$$

The lower index is usually omitted.

Some basic properties follow directly from the definition:

- The $\operatorname{Ext}^i(-, M)$ are contravariant for all $i \geq 0$.
- $\forall M, N \in \mathfrak{Mod}(R) : \quad \operatorname{Ext}^0(M, N) = \operatorname{Hom}_R(M, N)$
- For any short exact sequence of of R-modules

  $$0 \to A \to B \to C \to 0$$

  there is a long exact sequence

  $$\begin{aligned} 0 \to \operatorname{Ext}^0(C, M) \to \operatorname{Ext}^0(B, M) \to \operatorname{Ext}^0(A, M) \\ \to \operatorname{Ext}^1(C, M) \to \operatorname{Ext}^1(B, M) \to \operatorname{Ext}^1(A, M) \to \cdots . \end{aligned}$$

Now let X be a ringed space. The same definition as above along with the basic properties applies to the category of $\mathcal{O}_X$-modules as well. However in $\mathfrak{Mod}(\mathcal{O}_X)$ there are two notions of Hom-functor.

Definition A.2. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The right derived functors of the contravariant, left-exact

$$\operatorname{Hom}_{\mathcal{O}_X}(-, \mathcal{F}) : \mathfrak{Mod}(\mathcal{O}_X) \to \mathfrak{Mod}(k)$$

are denoted $\operatorname{Ext}^i_{\mathcal{O}_X}(-, \mathcal{F})$. They are referred to as **global Ext-functors**.

In contrast there is the **internal Hom-functor**

$$\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{F}) : \mathfrak{Mod}(\mathcal{O}_X) \to \mathfrak{Mod}(\mathcal{O}_X),$$

which is defined as

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})(U) := \mathrm{Hom}_{\mathcal{O}_X}\big(\mathcal{G}(U), \mathcal{F}(U)\big).$$

It is again contravariant and left-exact. The right derived functors are denoted by $\mathcal{E}xt^i_{\mathcal{O}_X}(-, \mathcal{F})$ and are called **local Ext-functors**.

In either case the lower index will be omitted when no confusion is expected.

We will now collect some more properties which we will need to perform computations involving either of the Ext-functors.

Proposition A.3. *Let X be a scheme and let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be $\mathcal{O}_X$-modules.*

a) $\forall i \geq 0 : \quad \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{F}) \cong H^i(X, \mathcal{F}).$

b) If $\mathcal{L}$ is locally finitely free then

$$\forall i \geq 0 : \quad \mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}).$$

Proof. a) [Mur06a, proposition 54].

b) [Mur06a, proposition 59]. □

Proposition A.4. *a) Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be an isomorphism of ringed spaces and $\mathcal{F}$ and $\mathcal{G}$ sheaves of modules on X. Then*

$$\forall i \geq 0 : \quad f_*\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G}) \cong \mathcal{E}xt^i_Y(f_*\mathcal{F}, f_*\mathcal{G}).$$

b) Let X be a Noetherian scheme, $\mathcal{F}$ and $\mathcal{G}$ sheaves of modules on X with $\mathcal{F}$ coherent. Then

$$\forall i \geq 0 \text{ and } x \in X : \quad \mathcal{E}xt^i(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x).$$

c) For $\mathcal{L}$ locally finitely free and $\mathcal{G}$ any sheaf of $\mathcal{O}_X$-modules it holds $\forall i > 0 : \mathcal{E}xt^i(\mathcal{L}, \mathcal{G}) = 0$.

Proof. [Mur06a, proposition 60, proposition 63, and corollary 56]. □

Appendix B

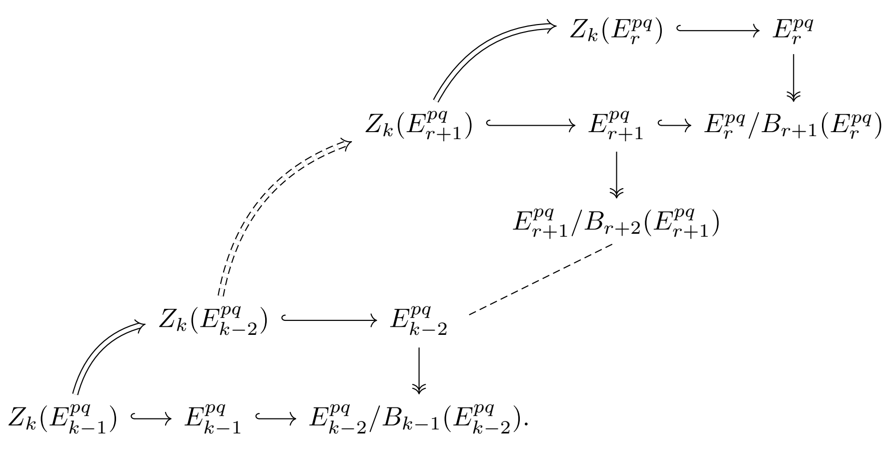
Spectral Sequences

A spectral sequence is an array of data. Roughly speaking it consists of:

1. for $p, q \in \mathbb{Z}$ and $r \in \mathbb{Z}_{\geq 0}$ objects E_r^{pq} in a fixed Abelian category,
2. morphisms $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ satisfying the property of a complex: $d_r^{p+r,q-r+1} \circ d_r^{pq} = 0$. The arrow directions are visualized in figure B.1. Notation:
$$\begin{aligned} Z_{r+1}(E_r^{pq}) &:= \ker d_r^{pq} \\ B_{r+1}(E_r^{pq}) &:= \operatorname{Im} d_r^{p-r,q+r-1}. \end{aligned}$$
3. And limit objects E_∞^{pq} given as quotient
$$E_\infty^{pq} = Z_\infty^{pq} / B_\infty^{pq},$$
where $B_\infty^{pq} \subseteq Z_\infty^{pq} \subseteq E_0^{pq}$ are some subobjects.

For fixed r the doubly indexed family $(E_r^{pq})_{p,q}$ is called r-th **page**. It is required that every page is the cohomology of the previous one:
$$E_{r+1}^{pq} \cong Z_{r+1}(E_r^{pq}) / B_{r+1}(E_r^{pq}). \tag{B.1}$$

Define objects $Z_k(E_r^{pq})$ and $B_k(E_r^{pq})$ for $k \geq r+1$ by successively pulling back along surjections:



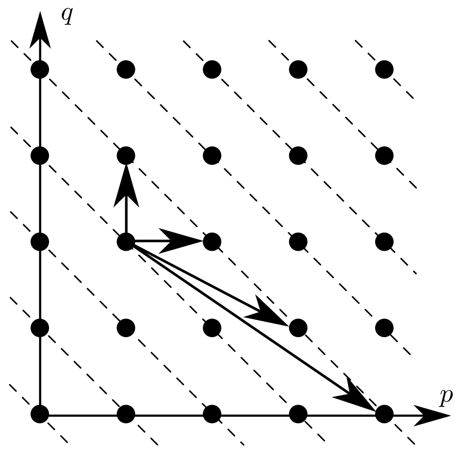


Figure B.1: Visualization of the p-q-plane. The dashed lines indicate loci with $p+q=n$, the arrows indicate the direction of the d_r^{pq} morphisms for $r=1,2,3,4$. Note how the direction of the maps moves clockwise with increasing r but always maps to the next dashed line.

Through careful observation one can see that for all $p,q\in\mathbb{Z}$ and $r\geq 0$ it holds

$$\forall k\geq r+1:\quad E_k^{pq}\cong Z_k(E_r^{pq})/B_k(E_r^{pq})$$

and

$$0\subseteq B_{r+1}(E_r^{pq})\subseteq B_{r+2}(E_r^{pq})\subseteq\cdots$$
$$\cdots\subseteq Z_{r+2}(E_r^{pq})\subseteq Z_{r+1}(E_r^{pq})\subseteq E_r^{pq}.$$

Definition B.1. The spectral sequence is said to be **weakly convergent** if

$$Z_\infty(E_0^{pq})=\inf_k Z_k(E_0^{pq})\text{ and}$$
$$B_\infty(E_0^{pq})=\sup_k B_k(E_0^{pq}).$$

It is said to be **biregular** if it is weakly convergent and the infimum and supremum are attained. In this case we write $E_r^{pq}\implies E_\infty^{pq}$. In practice only the term for $r=2$ on the left hand side is written.

For the complete definition and basic constructions of spectral sequences see [Mur06b].

Definition B.2. A spectral sequence is said to **degenerate on page** r if

$$\forall p,q\in\mathbb{Z}:d_r^{pq}=0.$$

In particular if a biregular sequence degenerates on page r we have by condition (B.1)

$$\forall s\geq r\text{ and }p,q\in\mathbb{Z}:E_s^{pq}\cong E_r^{pq}.$$

For our purposes it is most important to observe that we get an isomorphism $E_r^{pq}\cong E_\infty^{pq}$ in this case.

Definition B.3. A spectral sequence is called **first quadrant spectral sequence** if

$$\forall p < 0,\ q < 0 \text{ and } r \in \mathbb{Z} : E_r^{pq} = 0.$$

The following proposition gives a relation between local and global Ext functors via a spectral sequence [Mur06b, proposition 14].

Proposition B.4. *Let X be a ringed topological space and $\mathcal{F}$ and $\mathcal{G}$ sheaves of $\mathcal{O}_X$-modules. There is a biregular first quadrant spectral sequence*

$$E_2^{pq} := H^p\big(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})\big) \Longrightarrow \operatorname{Ext}^{p+q}(\mathcal{F}, \mathcal{G}),$$

called ***local-to-global Ext spectral sequence****.*

Remark B.5. We would like to highlight one particular special case of the corollary: assume X is a curve, i.e. one-dimensional. In this case the local-to-global Ext spectral sequence degenerates on the second page. The reason for this is that E_2^{pq} is non-trivial only for $p \in \{1, 2\}$ and thus the morphisms $d_2^{pq} : E_2^{pq} \to E_2^{p+2,q-1}$ are necessarily all constant zero. By convergence we get an isomorphism into the limit page:

$$\begin{aligned} \forall n \in \mathbb{N}_{\geq 0} : \quad \operatorname{Ext}^n(\mathcal{F}, \mathcal{G}) &\cong \bigoplus_{p+q=n} H^p\big(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})\big) \\ &\cong H^0\big(X, \mathcal{E}xt^n(\mathcal{F}, \mathcal{G})\big) \oplus H^1\big(X, \mathcal{E}xt^{n-1}(\mathcal{F}, \mathcal{G})\big). \end{aligned}$$

In general this isomorphism would be only up to grading, so we really need the fact that we are working in the category of k-vector spaces.

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