Goethe-Universität Frankfurt
Institut für Mathematik
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Komplexe Algebraische Geometrie
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Übungsblatt 9

Aufgabe 1 (7 Punkte)

The aim of this exercise is to show concretely in some examples that blow-ups can be used to resolve singularities of curves.
Let $\pi : \mathrm{Bl}_0(\mathbb{C}^2) \to \mathbb{C}^2$ be the blow-up of $\mathbb{C}^2$ at the origin and let $E \subset \mathrm{Bl}_0(\mathbb{C}^2)$ be the exceptional divisor. Recall that $\mathrm{Bl}_0(\mathbb{C}^{n+1})$ can be described as the subset of $\mathbb{P}^n \times \mathbb{C}^{n+1}$ satisfying the equations $z_i \cdot x_j = z_j \cdot x_i$ for $i, j = 0, \dots, n$, where $(z_0, \dots, z_n) \in \mathbb{C}^{n+1}$ and $(x_0 : \dots : x_n) \in \mathbb{P}^n$.

(a) Let $C_1 \subset \mathbb{C}^2$ be the reducible curve defined by $z_0 \cdot z_1 = 0$. Show that the closure of $\pi^{-1}(C_1 \backslash \{0\}) \subset \mathrm{Bl}_0(\mathbb{C}^2) \backslash E$ in $\mathrm{Bl}_0(\mathbb{C}^2)$ is a smooth curve. Describe it geometrically. Show that the points of E correspond to the lines through the origin in $\mathbb{C}^2$.

(b) Let $C_2 \subset \mathbb{C}^2$ be the irreducible singular curve defined by $z_0^2 - z_1^2(z_1 + 1) = 0$. Is the closure of $\pi^{-1}(C_2 \backslash \{0\}) \subset \mathrm{Bl}_0(\mathbb{C}^2) \backslash E$ in $\mathrm{Bl}_0(\mathbb{C}^2)$ a smooth curve?
Hint: we have already studied this curve, called *nodal cubic*, in Übungsblatt 2 Aufgabe 1. It is singular only in $(0,0) \in \mathbb{C}^2$, and we found the following parametrization by studying all lines passing through the origin in $\mathbb{C}^2$

$$\begin{aligned} \varphi : \mathbb{C} &\longrightarrow \mathbb{C}^2 \\ t &\longmapsto (t^3 - t, t^2 - 1). \end{aligned}$$

(c) Let C_3 be the singular curve defined by $(z_0 - z_1)\big(z_0^2 - z_1^2(z_1 + 1)\big) = 0$. Show that the closure of $\pi^{-1}(C_3 \backslash \{0\}) \subset \mathrm{Bl}_0(\mathbb{C}^2) \backslash E$ in $\mathrm{Bl}_0(\mathbb{C}^2)$ is a singular curve having only one singularity at a point $P \in E$. What happens if you repeat the same construction with another blow-up at P, i.e. consider the blow up $\mathrm{Bl}_P\big(\mathrm{Bl}_0(\mathbb{C}^2)\big)$?

Aufgabe 2 (2 Punkte)

Consider the $\mathbb{Z}/2\mathbb{Z}$-action $z \mapsto -z$ on $\mathbb{C}^2$ which has one fixed point and whose quotient $\mathbb{C}^2/\pm$ is not smooth.

(a) Show that the action lifts to a unique $\mathbb{Z}/2\mathbb{Z}$-action on the blow-up $\mathrm{Bl}_0(\mathbb{C}^2)$ and that this action is not free. What is the fixed locus?

(b) Prove that the quotient $X := \mathrm{Bl}_0(\mathbb{C}^2)/\pm$ is a manifold.
Hint: the action is not free, so we can not use the results for quotients under actions of discrete groups. However, it is possible to construct explicitly a holomorphic atlas of X. To do that, write down the charts of $\mathrm{Bl}_0(\mathbb{C}^2)$ and modify them to find charts of X.

Aufgabe 3 (2 Punkte)

Let $\hat{X} \to X$ be the blow-up of a surface X at a point $x \in X$. Show that the pull-back of sections defines an isomorphism $H^0(X, K_X) \cong H^0(\hat{X}, K_{\hat{X}})$.
Hint: see Übungsblatt 2 Aufgabe 2 and Übungsblatt 5 Aufgabe 4, both proved using Hartogs' Extension Theorem.

Aufgabe 4 (5 Punkte)

Let X be a connected compact manifold, $x \in X$ a point and $\hat{X} := \mathrm{Bl}_x(X)$ the blow-up of X at x. Show that the line bundle $\mathcal{O}_{\hat{X}}(E)$, where E is the exceptional divisor, admits (up to scaling) only one section.
Hint: show that every section $s \in H^0(\hat{X}, \mathcal{O}_{\hat{X}}(E))$ is such that $s|_E \equiv 0$. Recall that the divisors associated to two non-trivial sections s, t are equal, i.e. $Z(s) = Z(t)$, if and only if there is a map $f \in \mathcal{O}^*(\hat{X})$ such that $t = f \cdot s$.

Abgabe Zu Beginn der Vorlesung um **10:00** am **Montag, den 14. Januar**.