Goethe-Universität Frankfurt
Institut für Mathematik
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Komplexe Algebraische Geometrie
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Übungsblatt 8

Aufgabe 1 (4 Punkte)

Let L_1 and L_2 be holomorphic line bundles on complex manifolds X_1 and X_2, respectively. To two sections $s_i \in H^0(X_i, L_i)$, $i = 1, 2$, one associates a section

$$s_1 \cdot s_2 \in H^0(X_1 \times X_2, p_1^*(L_1) \otimes p_2^*(L_2)).$$

Show that if $s_1^1, \dots, s_1^k$ ans $s_2^1, \dots, s_2^\ell$ are linearly independent sections of L_1 and L_2, respectively, then $s_1^i s_2^j$ form linearly independent sections of $p_1^*(L_1) \otimes p_2^*(L_2)$. Here, p_1 and p_2 are the two projections $p_i : X_1 \times X_2 \to X_i$.
Hint: suppose that $s_1^i s_2^j$ are linearly dependent and work on suitable restrictions of $X_1 \times X_2$.

Aufgabe 2 (3 Punkte)

Let $P := (0 : \dots : 0 : 1) \in \mathbb{P}^n$ and consider the linear system

$$\left\{ s \in H^0\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)\big) \;:\; s(P) = 0 \right\}.$$

(a) Show that it defines a holomorphic map $\varphi : \mathbb{P}^n \backslash \{P\} \to \mathbb{P}^{n-1}$, and that this is actually a line bundle over $\mathbb{P}^{n-1}$.

(b) Show that the previous line bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$.

(c) What happens if one chooses $P \neq (0 : \dots : 0 : 1)$?

Aufgabe 3 (2 Punkte)

Are there holomorphic vector fields on $\mathbb{P}^n$, i.e. global sections of $T_{\mathbb{P}^n}$, which vanish only in a finite number of points? If yes, in how many?
Hint: use Aufgabe 3 Übungsblatt 7 and Aufgabe 2 Übungsblatt 6.

Aufgabe 4 (7 Punkte)

Let $E \to X$ be a holomorphic vector bundle and let $s \in H^0(X, E)$ its zero section. On the complement $E \backslash s(X)$ one has a natural $\mathbb{C}^*$-action. The quotient

$$\mathbb{P}(E) := (E \backslash s(X)) / \mathbb{C}^*$$

is a complex manifold that admits a holomorphic projection $\pi : \mathbb{P}(E) \to X$ such that $\pi^{-1}(x)$ is isomorphic to $\mathbb{P}(E(x))$. We call $\mathbb{P}(E)$ the *projective bundle associated to* E or, simply, the *projectivization of* E.
The surface $\Sigma_n := \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)\big)$ is called the n-th *Hirzebruch surface*. Show that Σ_n

is isomorphic to the hypersurface $V := Z(x_0^n y_1 - x_1^n y_2) \subset \mathbb{P}^1 \times \mathbb{P}^2$, where $(x_0 : x_1)$ and $(y_0 : y_1 : y_2)$ are the homogeneous coordinates of $\mathbb{P}^1$ respectively $\mathbb{P}^2$.
Hint: write down the coordinates of the points in $V \cap U_0 \times \mathbb{P}^2$ and $V \cap U_1 \times \mathbb{P}^2$. Use these to construct explicitly two maps:

$$\begin{aligned} \phi_{U_j} : \Sigma_n|_{U_j} \cong U_j \times \mathbb{P}^1 &\to \mathbb{P}^1 \times \mathbb{P}^2 \\ ((x_0 : x_1), (a : b)) &\mapsto ((x_0 : x_1), (? :? :?)) \end{aligned}$$

such that they glue together to an injective map $\phi : \Sigma_n \to \mathbb{P}^1 \times \mathbb{P}^2$ whose image is V.

Abgabe Zu Beginn der Vorlesung um **10:00** am **Montag, den 17. Dezember**.