# Faithful Tropicalization of the Grassmannian of planes

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# Motivation

**Goal:**

The Grassmannian of planes is faithfully tropicalized by the Plücker embedding, i.e. the tropical Grassmannian of planes is homeomorphic to a closed subject of the Berkovich analytic Grassmannian.

**Motivation:**

Faithful tropicalization results for curves by Baker, Payne and Rabinoff.

# Setting

$K$ non-Archimedean complete field with respect to $|\ |_K$.

**Example:**

$\mathbb{Q}_p$, finite extensions of $\mathbb{Q}_p, \mathbb{C}_p$, Laurent series or Puiseux series, any field with the trivial valuation.

$X/K$ variety
$X^{an}$ Berkovich analytic space.
If $X = Spec\ A$ affine, then
$X^{an} = \{$ multiplicative seminorms $A \to \mathbb{R}_{\geqq 0}$ extending $|\ |_K\}$
equipped with the topology of pointwise convergence.

**Note:** Every $K-$rational point $a \in X(K)$ induces a point in $X^{an}$ by $A \to \mathbb{R}_{\geqq 0}, f \mapsto | f(a) |_K$ .

# Example: Berkovich affine line

Let $X = \mathbb{A}^1_K = \mathrm{Spec} K[x]$ and assume $K$ algebraically closed. Put $D(a,r) = \{x \in K : \mid x - a \mid \leqq r\} \subset K = \mathbb{A}^1_K(K)$. Then $X^{an}$ consists of the following points (seminorms): Let $f \in K[x]$.

**Points of type 1:** $\mid f \mid_a = \mid f(a) \mid$ for $a \in K = \mathbb{A}^1_K(K)$.

**Points of type 2:** $\mid f \mid_{a,r} = \sup_{x \in D(a,r)} \mid f(x) \mid$ for a disc $D(a,r)$ with $r \in \mid K^\times \mid$

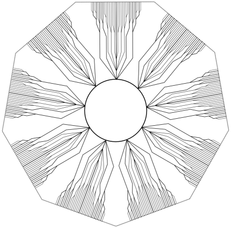
**Points of type 3:** $\mid f \mid_{a,r} = \sup_{x \in D(a,r)} \mid f(x) \mid$ for a disc $D(a,r)$ with $r \notin \mid K^\times \mid$

**Points of type 4:** $\mid f \mid_{a,r} = \varinjlim_{n \to \infty} \mid f \mid_{a_n, r_n}$ for a nested sequence $D(a_1, r_1) \supset D(a_2, r_2) \ldots$ of discs.

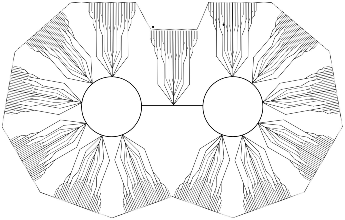
# Example: Berkovich affine line

The topological space $(\mathbb{A}^1_K)^{an}$ can be identified with an $\mathbb{R}$-tree which has infinitely many branches growing out of every point contained in a dense subset of any line segment.
Branching occurs at points of type 2. The leaves of the tree are type 1 or type 4 points.
In order to get the (compact) Berkovich projective line $(\mathbb{P}^1_K)^{an}$ it suffices to add one leaf $\infty$ of type 1.

# Berkovich curves (by Till Wagner)



Tate curve



Genus two curve

# Berkovich spaces in higher dimensions

**!** The description of Berkovich curves as $\mathbb{R}-$trees relies on the fact that $K[x]$ is a factorial ring.

In dimension $\geqq 2$, there is no explicit description of the points in Berkovich spaces.

$$\begin{array}{rcl} \text{Trop: } (\mathbb{G}_m^r)^{an} & \longrightarrow & \mathbb{R}^r \\ \gamma & \longmapsto & \big(\log\gamma(x_1),\ldots,\log\gamma(x_r)\big) \end{array}$$

$X_0 \subset X$ very affine, i.e. closed embedding $\varphi : X_0 \hookrightarrow \mathbb{G}_m^r$.
Associated tropicalization:

$$\text{trop: } X_0^{an} \overset{\varphi^{an}}{\hookrightarrow} (\mathbb{G}_m^r)^{an} \overset{\text{Trop}}{\rightarrow} \mathbb{R}^r.$$

More generally: $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}, \underline{1} = (1,\ldots,1)$

$$\begin{array}{rcl} \text{Trop: } (\mathbb{P}_K^r)^{an} & \longrightarrow & \big(\overline{\mathbb{R}}^{r+1} \backslash \{-\infty \cdot \underline{1}\}\big) \big/ \mathbb{R}\underline{1} = \mathbb{TP}^r \\ \gamma & \longmapsto & \big(\log\gamma(x_o),\ldots,\log\gamma(x_r)\big) + \mathbb{R}\underline{1} \end{array}$$

$\varphi : X \hookrightarrow \mathbb{P}_K^r$ gives rise to tropicalization:

$$\text{trop } : X^{an} \hookrightarrow (\mathbb{P}_K^r)^{an} \rightarrow \mathbb{TP}^r$$

# Results on curves by Baker-Payne-Rabinoff

Assume that $\dim X = 1$ and $|\ |_K$ non-trivial.

**Theorem ([BPR])**

For every finite subgraph $\Gamma$ of $X^{an} \setminus \{\text{leaves}\}$ there exists a tropicalization mapping $\Gamma$ isometrically to its image.

**Theorem ([BPR])**

If $\Gamma' \subset \mathrm{Trop}(X_0)$ for some very affine $X_0 \subset X$ is a finite subgraph with tropical multiplicity one everywhere, there exists a subgraph $\Gamma \subset X^{an}$ mapping isometrically to $\Gamma'$ under trop.

# Higher dimensions

**Higher-dimensional varieties?**

- In higher dimensions, there is no polyhedral description of Berkovich spaces generalizing the $\mathbb{R}-$tree structure of analytic curves.
- in general there are no semistable models (which play an important role in [BPR]).
- metrics have to be replaced with piecewise linear structures.

**Example:**

Grassmannian

# Grassmannian

$Gr(d,n)$ Grassmannian of $d-$dimensional subspaces of an $n-$dimensional vector space.

Plücker embedding

$$\begin{array}{rccc} \varphi : & Gr(d,n) & \hookrightarrow & \mathbb{P}_K^{\binom{n}{d}-1} \\ & \omega & \mapsto & \wedge^d \omega \end{array}$$

Tropical Grassmannian:

$$\text{trop}: \; Gr(d,n)^{an} \overset{\varphi^{an}}{\hookrightarrow} (\mathbb{P}_K^{\binom{n}{d}-1})^{an} \overset{\text{Trop}}{\to} \mathbb{TP}^r$$

$\mathcal{T}Gr(d,n) =$Image (trop).

# Tropical Grassmannian

$$Gr_0(d,n) = \varphi^{-1}(\mathbb{G}_m^{\binom{n}{d}}/\mathbb{G}_m)$$

$$\mathcal{T}Gr_0(d,n) = \text{Image}\left(\mathrm{trop}\,|_{Gr_0(d,n)}\right)$$

$\mathcal{T}Gr_0(d,n)$ is the tropical Grassmannian $\mathcal{G}'_{d,n}$ investigated by Speyer and Sturmfels 2004.

**Theorem (Speyer-Sturmfels)**

$\mathcal{T}Gr_0(d,n)$ is a fan of dimension $d(n-d)$. It contains a lineality space of dimension $n-1$.

**Theorem (Speyer-Sturmfels)**

$\mathcal{T}Gr_0(2,n)$ is the space of phylogenetic trees.

From now on: $d = 2$

Plücker coordinates on $\mathbb{P}_K^{\binom{n}{2}-1}$ : $p_{ij}$ for all $\{i, j\} \subset \{1, \ldots, n\}; i \neq j$.
Plücker relations

$$p_{ij}p_{kl} = p_{ik}p_{jl} - p_{il}p_{jk}$$

for $i, j, k, l$ pairwise distinct.

# Phylogenetic trees

A phylogenetic tree on $n$ leaves is a tree $T$ whose leaves are labelled $1, 2, \ldots, n$, which is endowed with a weight function

$$\omega : \text{edges}\,(T) \to \mathbb{R}.$$

For every phylogenetic tree $(T, \omega)$ and all $i, j \in \{1, \ldots, n\}$ let

$$x_{ij} = \text{sum of weights along the path from } i \text{ to } j.$$

# Phylogenetic trees

Then $x_{ij}$ satisfy the tropical Plücker relation:
For all pairwise distinct $i, j, k, l \in \{1, \dots, n\}$ the maximum among

$$x_{ij} + x_{kl}, x_{ik} + x_{jl}, x_{il} + x_{jk}$$

is attained at least twice.

Hence $(x_{ij})_{ij}$ is a point in the tropical Grassmannian $\mathcal{T}Gr_0(2, n)$.

# Main result

**Theorem (Cueto, Häbich, Werner 2013)**

There exists a continuous section $\sigma : \mathcal{T}Gr(2, n) \to Gr(2, n)^{an}$ of the tropicalization map.
Hence the tropical Grassmannian of planes is homeomorphic to a closed subset of the Berkovich analytic Grassmannian.

**Motivation:**

$\mathcal{T}Gr_0(2, n)$ has tropical multiplicity one everywhere by Speyer/Sturmfels. On compact subsets of tropical curves this implies existence of a section by [BPR].

**First idea**

Fix $i \neq j$. $U = \varphi^{-1}\{p_{ij} \neq 0\}$ big cell.

Write $u_{kl} = \frac{p_{kl}}{p_{ij}}$ for the affine cordinates on $\{p_{ij} \neq 0\}$.
Then $U \simeq \mathbb{A}_K^{2(n-2)}$ with respect to $(u_{ik}, u_{jk})_{k \neq i,j}$.

Skeleton map

$$\begin{array}{rcl} \overline{\mathbb{R}}^{2(n-2)} & \hookrightarrow & (\mathbb{A}_K^{2(n-2)})^{an} \\ r = (r_1, \ldots, r_{2(n-2)}) & \mapsto & \gamma_r, \\ \gamma_r(\sum_I a_I u^I) & = & \max_I\{|a_I| \exp(\langle r, I\rangle)\} \end{array}$$

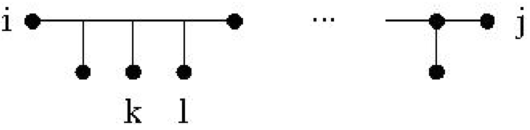
$\big(\gamma_r$ is a norm on the polynomial ring, $u = (u_{ik} u_{jk})_{k \neq i,j}\big)$.
In particular, writing $r = (r_{ik}, r_{jk})_{k \neq i,j}$,

$$\begin{array}{rcl} \gamma_r(u_{kl}) & = & \max\{\exp(r_{ik} + r_{jl}), \exp(r_{jk} + r_{il})\} \\ & = & \exp(\max\{r_{ik} + r_{jl}, r_{jk} + r_{il}\}). \end{array}$$

This provides a section of trop: $Gr(2,n)^{an} \to \mathcal{T}Gr(2,n)$ on all $x \in \mathcal{T}Gr(2,n)$ satisfying

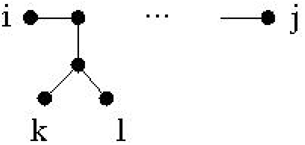
$$x_{kl} + x_{ij} = \max\{x_{ik} + x_{jl}, x_{jk} + x_{il}\}.$$

Ok, if $x$ comes from a phylogenetic caterpillar tree




Hence this proves our claim for $n = 4$.

Not ok, if $x$ comes from a tree of the form



# The proof

In general, the definition of our section depends

1. on the combinatorial type of the phylogenetic tree
2. on the position in the coordinate hyperplane arrangement in $\mathbb{A}_K^{2(n-2)}$.

Strategy:

$T$ combinatorial $n-$labelled tree.

| | |
|---|---|
| $C_T \subset \mathcal{T}Gr_0(2,n)$ | set of points corresponding to $(T,\omega)$ for some weight function $\omega$. |
| $\overline{C_T} \subset \mathcal{T}Gr(2,n)$ | closure. |

**Note:**

$$\mathcal{T}Gr(n,n) = \overline{\mathcal{T}Gr_0(2,n)}.$$

$\Sigma$ Stratum in coordinate hyperplane arrangement in $U \simeq \mathbb{A}_K^{2(n-2)}$.

**Theorem**

Depending on $\Sigma, U, T$ there exists a set $I \subset \{u_{kl} : k \neq l\}$ of cardinality $2(n-2)$ which generates the function field $K(U)$, such that

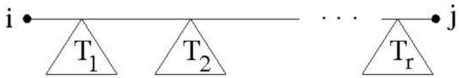
$$\overline{\mathbb{R}}^{2(n-2)} \underset{\text{Skeleton map}}{\hookrightarrow} \text{Spec } K[I]^{an} \underset{\text{rational}}{\dashrightarrow} U^{an}$$

is a section of trop over $\Sigma^{an} \cap \overline{C}_T$.

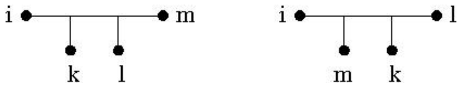
**Example:**

$\Sigma =$ complement of all coordinate hyperplanes in $U$.

Arrange $T$ as




with subtrees $T_1, \dots, T_r$.
Order the leaves of each $T_j$ such that the cherry property holds:
For leaves $k \prec l \prec m$ either $\{kl\}$ or $\{lm\}$ is a cherry in $\{i, k, l, m\}$ (and similary for $\{k, l, m, j\}$).



# The proof: example

Write $s_1 \prec s_2 \prec \ldots \prec s_p$ for the leaves in $T_1$.

Take variables

$$u_{is_1}, u_{is_2}, \ldots, u_{is_p},$$
$$u_{js_1}, u_{s_1 s_2}, \ldots, u_{s_{p-1} s_p}$$

on $T_1$.

$I$ = union of these sets of variables over all subtrees $T_1, \ldots, T_r$.

# The proof: what remains to be done

These local sections are independent of all choices and glue. Hence we get the desired section

$$\sigma : \mathcal{T} Gr(2, n) \longrightarrow Gr^{an}(2, n)$$

of the tropicalization map.

Technical lemma: show that $\sigma$ is continuous.

# Shilov boundaries

**Proposition**

The section $\sigma : \mathcal{T}Gr(2,n) \to Gr(2,n)^{an}$ associates to $x$ the unique Shilov boundary point of the affinoid domain $\text{trop}^{-1}(\{x\}) \subset Gr(2,n)^{an}$.

What does this mean? Assume $\sigma(x)$ is contained in the analytification of the big cell $U_{ij} = \mathrm{Spec}(R_{ij})$ in the Grassmannian. Then $\sigma(x)$ is a multiplicative seminorm on $R_{ij}$ extending the absolute value on $K$. For all other such multiplicative seminorms $\gamma$ satisfying

$$\mathrm{trop}(\gamma) = x$$

we have

$$\gamma(f) \leq \sigma(x)(f)$$

for all $f \in R_{ij}$.

# Tropical multiplicities

**Proposition**

For all $x \in \mathcal{T}Gr(2, n)$ the tropical multiplicity of $x$ (in the ambient torus) is one.

(For $x \in \mathcal{T}Gr_0(2, n)$ this is due to Speyer/Sturmfels.)

**Proof:** Use the local coordinate system $I$ to calculate the initial degeneration.

# Algebraic consequence

**Corollary**

Let $J$ be a subset of the Plücker coordinates not containing $p_{ij}$. Then $Gr_J(2,n) = \varphi^{-1}\{p_{kl} = 0 \Leftrightarrow p_{kl} \in J\}$ is an affine variety with coordinate ring

$$K[u_{kl}^{\pm 1} : u_{kl} \in I \backslash J]_S,$$

where $S$ is the multiplicative subset generated by all $u_{kl} \notin J$.

Sturmfels: The spherical complex associated to the tropical Grassmannian $\mathcal{T}Gr_0(2,5)$ is the Petersen graph.

**Corollary**

Our section induces an embedding of the Petersen graph into the quotient $(Gr_0(2,5)/\mathbb{G}_m^5)^{an}$ of the analytic very affine Grassmannian $Gr_0(2,5)^{an}$.