BRIDGELAND STABILITY CONDITIONS AND APPLICATIONS

GAUS AG-DaFra-Seminar – SoSe 2023

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Introduction

The notion of stability conditions on triangulated categories was introduced by Tom Bridgeland in [Bri07] as a way to understand the geometry behind the myriad notions of stability which appear in the construction of various types of moduli spaces. It has its roots in physics and was inspired by the work of M. Douglas on Π-stability for D-branes in string theory (cf. [Bri09, Section 2]).

In a triangulated category $\mathbf{D}$, such as the derived category of coherent sheaves on a variety, objects can be quite complicated and difficult to classify. Bridgeland stability conditions provide a way to organize these objects into semistable and stable components based on numerical invariants. Its definition comes with many nice features, for example:

(i) the most interesting one is that the set of stability conditions $\mathrm{Stab}(\mathbf{D})$ has the structure of a complex manifold;
(ii) the choice of a stability condition picks out classes of stable objects for which one can hope to form well-behaved moduli spaces;
(iii) the space of all stability conditions $\mathrm{Stab}(\mathbf{D})$ allows one to bring geometric methods to bear on the problem of understanding t-structures on $\mathbf{D}$;
(iv) the space $\mathrm{Stab}(\mathbf{D})$ provides a complex manifold on which the group $\mathrm{Aut}(\mathbf{D})$ naturally acts.

It is therefore not surprising that Bridgeland stability conditions have since become a powerful tool for understanding the geometry of moduli spaces and related structures in a variety of mathematical contexts, including algebraic geometry, representation theory, and mathematical physics; and has also become central in modern homological algebra and led to new connections between these fields.

The goal of this seminar is to discuss (i) and (ii). More specifically, in the first half of the seminar, we give a gentle introduction to Bridgeland stability conditions and establish its foundational basis, leading to the proof of Bridgeland's deformation theorem:

Theorem 0.1 ([Bri07, Theorem 1.2])**.** *The space of stability conditions on a triangulated category is a complex manifold.*

In the second half of the seminar, we discuss (ii) for $\mathbf{D} = \mathbf{D}^b(X)$ the bounded derived category of coherent sheaves on a smooth projective complex algebraic variety X. We will confine ourselves to the case where X is a surface as there are only a few sporadic examples in higher dimensions so far [1]. And at the end, we will further restrict ourselves to the case where X is a $K3$ surface, showing the existence of coarse moduli space of Bridgeland semistable objects with fixed numerical invariants and studying its positivity.

Theorem 0.2 ([BM14, Theorem 1.3])**.** *Let σ be a generic Bridgeland stability condition on a $K3$ surface X. Then the coarse moduli space $M_\sigma(v)$ of σ-semistable objects with Mukai vector $v \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ on X exists as a normal projective variety with $\mathbb{Q}$-factorial singularities.*

And last but not least, it is worth mentioning that many interesting new connections between Bridgeland stability conditions and other mathematical objects have recently emerged, such as

- stability conditions and differential equations (cf. [BTL12]);
- stability conditions and quadratic differentials (cf. [BS15], [BMQS22]);
- Bridgeland stability over non-archimedean fields (see Kontsevich's IHES lectures [Kon21]);
- $\cdots$

Talks

Time, Format, and Places.

Time: Thursdays (20.4, 4.5, 25.5, 1.6, 22.6, 29.6) at 15:15 - 17:45 CEST.

Format: hybrid – in person but will stream simultaneously via Zoom.

Places: Frankfurt: Room 711 Groß, Robert-Mayer-Straße 10.

What does $\star$, $\star\star$, and $\star\star\star$ mean?

$\star$: Suitable for Masters- or Ph.D. students without much background in complex algebraic geometry; talks should be straightforward to prepare.
$\star\star$: Suitable for Ph.D. students and postdocs; usually requires a background in complex algebraic geometry or knowledge of almost all previous talks.
$\star\star\star$: Suitable for ambitious Ph.D. students and postdocs, as well as for Professors. Requires a solid background in complex algebraic geometry and/or the willingness to engage with the material in significant depth.

[1] Computing Bridgeland stability conditions for specific examples is still very challenging, and many explicit examples are still unknown. For example, is $\mathrm{Stab}(X) \neq \emptyset$ for X a quintic threefold?

Mathematics is made together.

The goal of the GAUS-AG is not only to learn about advances in current research but also to get to know each other, discuss and participate actively. We strongly encourage you to ask the organizers and your colleagues about any problem arising in the preparation of the talks. Questions about the general state of the art and curiosities on the topic of the seminar are more than welcome.

Part I. Bridgeland stability conditions on triangulated categories

Talk 1: Introduction. (20.04)[J. Chen]

Talk 2: Triangulated categories. $\star$ (20.04)[Y. M. Wong]

Take a tour to the basics of triangulated categories by following [KS94, Sections 1.3 - 1.7] (we recommend [Tho01] for topological intuition): starting from an additive category $\mathcal{A}$, define the category of cochain complexes $\mathbf{C}^*(\mathcal{A}), * \in \{\emptyset, b, +, -\}$; construct the homotopy category $\mathbf{K}^*(\mathcal{A})$ by equating homotopy equivalent maps between complexes. The naturally defined shift functor and the construction of mapping cones equip $\mathbf{K}^*(\mathcal{A})$ with a collection of distinguished (or exact) triangles. Show that such a collection satisfies properties TR0–TR5 [KS94, Proposition 1.4.4]. Abstracting these properties, define what a triangulated category is [KS94, Definition 1.5.1] and deduce some basic properties from the axioms [KS94, Proposition 1.5.3, Corollary 1.5.5], [Huy06, Exercise 1.36, 1.38]. Now let $\mathcal{A}$ be an abelian category. Define the derived category $\mathbf{D}^*(\mathcal{A})$ of $\mathcal{A}$ by localizing $\mathbf{K}^*(\mathcal{A})$ by inverting quasi-isomorphisms via a calculus of fractions [KS94, Sections 1.6–1.7]. Show that $\mathbf{D}^*(\mathcal{A})$ is a triangulated category [KS94, Proposition 1.6.9].

[Option] If time allowed, briefly discuss some examples: some choices are

- $\star$ $\mathbf{D}^b(X) := \mathbf{D}^b(\mathrm{Coh}(X))$, *where* $\mathrm{Coh}(X)$ *is the category of coherent sheaves on a smooth projective variety* X*;*
- $\star$ $\mathbf{D}^b(Q) := \mathbf{D}^b(\mathrm{Rep}(Q))$, *where* $\mathrm{Rep}(Q)$ *is the category of representations of a quiver* Q [Bri12]*;*
- $\star$ $\mathbf{D}(A)$ *the derived category of (right) dg* A*-modules, where* A *is a dg algebra* [Kel94]*;*
- $\star\star$ $\mathbf{D}(\mathcal{S})$ *the stable homotopy category: it is a triangulated category but not of the form* $\mathbf{D}(\mathcal{A})$ *for an abelian category* $\mathcal{A}$ [Wei94, Section 10.9]*;*
- $\star\star\star$ $\mathbf{DM}_k$ *the triangulated category of Voevodsky's motives: conjecturally, it is the derived category of mixed* k*-motives.*

Talk 3: Derived categories of coherent sheaves. $\star$ (or $\star\star$)

(04.05)[**Y. Kleibrink**]

Give an overview of the bounded derived category $\mathbf{D}^b(X)$ of coherent sheaves on a smooth complex projective variety X as it is the main player in this seminar. We are going to follow the notes [Č05] but refer to [Huy06] for details. First, we discuss the basics of derived functors on $\mathbf{D}^b(X)$: define the right (resp. left) derived functor $\mathbf{R}\,F$ (resp. $\mathbf{L}\,F$) of a left (resp. right) exact functor $F\colon \mathcal{A} \to \mathcal{B}$ when $\mathcal{A}$ has enough injectives (resp. projectives) [Č05, definition 2.10 (for right derived functors)]; the key is [Č05, Corollary 2.7] about injective resolutions, which tells us to how to perform calculations in $\mathbf{D}^*(\mathcal{A})$ usually. Briefly discuss how to resolve the problems of defining $\mathbf{L}\,F$ (resp. $\mathbf{R}\,F$), when $\mathcal{A}$ doesn't have enough projectives [Č05, **2.4**] (resp. injectives (which is the case in $\mathrm{Coh}(X)$) [Č05, **2.8**] and [Huy06, Proposition 2.42]). Applying to $\mathbf{D}^b(X)$, define derived functors (e.g., $\otimes^{\mathbf{L}}, \mathbf{R}\,\mathcal{H}\mathrm{om}$; $\mathbf{L}\,f^*, \mathbf{R}\,f_*$ for a morphism $f\colon X \to Y$ between smooth projective varieties) in algebraic geometry [Č05, **2.5**] and *[Option] explain some compatibilities* [Č05, **2.7**]. *[Option] State that the total direct image functor $\mathbf{R}\,f_*$ has a right adjoint $f^!$ (Grothendieck-Verdier duality)* [Č05, Theorem 4.3]. Define the Serre functor and summarize its crucial properties [Č05, **4.6, 4.7**].

Second, define Fourier-Mukai transforms [Huy06, Definition 5.1]; show that many of the functors defined above are Fourier-Mukai transforms (with various kernels) [Huy06, Examples 5.4]. State the theorem [CS12, Theorem 3.1] of Orlov which says this is always the case under certain assumptions on the functors. Remark that the assumptions in the theorem can be weakened.

[Option] If time allowed, discuss either

- $\star\star$ *the reconstruction theorem* [Č05, Theorem 4.7] *of Bondal-Orlov for smooth projective varieties with ample or anti-ample canonical bundles; or*
- $\star\star$ *the semi-orthogonal decomposition of* $\mathbf{D}^b(\mathbb{P}^n)$ [Č05, Theorem 3.1].

Talk 4: Stability in abelian categories. $\star$

(04.05)[**A. Kuhrs**]

In order to understand stability conditions on triangulated categories it is enough to understand stability conditions on abelian categories (this talk) and tilting of hearts of bounded t-structures (see Talk 7). Motivate the definition of stability condition on an abelian category by first discussing the μ-stability of vector bundles on algebraic curves C [Bay, Subsection 2.1]: show the existence of Harder-Narasimhan filtration on coherent sheaves on C [Bay, Theorem 2.1.6]. Given an abelian category $\mathcal{A}$ (resp. a triangulated category $\mathbf{D}$), define its Grothendieck group $K_0(\mathcal{A})$ (resp. $K_0(\mathbf{D})$). Give the definition of a stability function on $\mathcal{A}$ [MS17, Definition 4.1]; then define (semi)stable objects in $\mathcal{A}$. Give (non-)examples of stability functions. Prove Schur's lemma [MS17, Lemma 4.5]. Give the definition of a stability condition on an abelian category [MS17, Definition 4.6]. Explain that the existence of Harder-Narasimhan filtrations is actually a rather weak assumption in the definition of stability functions [Bri07, Proposition 2.4]. Give the criterion for the existence of Harder-Narasimhan filtrations [MS17, Proposition 4.10].

Talk 5: Stability conditions on triangulated categories. $\star$ (25.05)[**K. Kühn**]

Define t-structure on a triangulated category $\mathbf{D}$ by following the notes [Kli, Definition 13.1.1] and show that the heart of a t-structure on $\mathbf{D}$ is abelian [Kli, Theorem 14.0.9 (1)]. Give the definition of a bounded t-structure [Huy14, Definition 4.13] and its equivalent description[Bri07, Lemma 3.2]. Define slicing of $\mathbf{D}$ [Bri07, Definition 3.3] and prove some of its basic properties, for example, see [Bay, Remark 4.1.2]. Give the definition of a stability condition on $\mathbf{D}$ [MS17, Definition 5.8]. Briefly explain the support property. Prove [MS17, Lemma 5.11]. Give some examples of stability conditions. Remark that it is often very difficult to construct stability conditions [Bay, Remark 4.2.1].

Talk 6: The stability manifold. $\star\star$ (25.05)[**J. Horn**]

Define the topological (in fact, generalized metric) space $\mathrm{Stab}(\mathbf{D})$ of stability conditions on a(n) (essentially small) triangulated category $\mathbf{D}$. State Bridgeland's deformation theorem [Bay, Theorem 5.1.1] and sketch a proof of it by following [Bay, Subsection 5.5] (see also [MS17, Theorem 5.15]). Define the actions (by isometries) of $\mathrm{Aut}(\mathbf{D})$ and $\widetilde{\mathbf{GL}^+(2,\mathbb{R})}$ on $\mathrm{Stab}(\mathbf{D})$. Describe the stability manifold $\mathrm{Stab}(C) := \mathrm{Stab}(\mathbf{D}^b(C))$ for C a smooth projective curve of positive genus [MS17, Example 5.17 (2)]. If time allowed, discuss also the stability manifold of $\mathbb{P}^1$ [Oka06] (see also [Bay, Subsection 5.4]).

Part II. Applications

Talk 7: Stability conditions on surfaces. $\star$ (01.06)[**R. Zuffetti**]

Introduce torsion pairs ([MS17, Definition 6.1]) and define tilting of hearts at torsion pairs ([MS17, Lemma 6.3 and Definition 6.6]). Then using these newly defined concepts, construct Bridgeland stability conditions on surfaces: state [MS17, Theorem 6.10] and [MS17, Theorem 6.13]. The latter theorem is essential in proving the former: provide an explanation of why this is the case.

Talk 8: Walls and chambers. $\star\star$ (01.06)[**J. So**]

Classify the hearts one can obtain by tilting $\mathrm{Coh}(X)$, see [Huy14, Theorem 3.2]. Then discuss wall and chamber structures for (α, β)-planes, see [BM14, Section 6.4] and state the structure theorem for walls on surfaces, see [BM14, Proposition 6.22].

Talk 9: Moduli spaces of Bridgeland semistable objects. ⋆⋆ (22.06)[**A. Bud**]

Start by providing some intuition for Artin (algebraic) stacks, see [Ols16, Chapter 8]. Discuss families of objects in the bounded derived category. In particular, Introduce the notion of a perfect object, as well as introduce the moduli space of Bridgeland semistable objects ([MS17, Definition 5.22]) and present the subsequent discussion.

Next, present [MS17, Theorem 6.30] of Toda realizing the moduli space of Bridgeland semistable objects as an Artin stack (cf. [Tod07, Theorem 1.3] and sketch its proof:

(1) the construction of a new heart on a product of varieties [AP06, Theorem 2.6.1] of Abramovich and Polishchuk (cf. [MS17, Theorem 8.2]);
(2) the lemma of Toda showing that being in the heart is an open property under certain assumptions, see [Tod07, Lemma 4.7].

Talk 10: The Positivity Lemma. ⋆⋆⋆ (22.06)[**J. Chen**]

Via the Donaldson morphism, one obtains a divisor class on the moduli space of Bridgeland semistable objects, see [BM14, Section 4], [BM14, Section 8.1]. State the Positivity Lemma, see [BM14, Lemma 3.3] and

(1) prove the first part of the lemma: "nefness of the divisor class";
(2) prove the "S-equivalence" part of the lemma: we describe under what circumstances the intersection of the divisor with a curve in the moduli can be 0.

Talk 11: The projectivity of the Bridgeland moduli space for K3. ⋆⋆⋆ (29.06)[**M. Möller**]

Prove Theorem 1.3 of [BM14]: following [BM14, Section 7], prove the existence of the coarse moduli space; furthermore, one can improve the Positivity Lemma of the previous talk in the case of $K3$ surfaces: show ampleness of the divisor, see [BM14, Corollary 7.5].

Talk 12: Bridgeland stability conditions on Calabi-Yau manifolds. (29.06)[**E. Macrì**]

I will give a short overview on Bridgeland stability conditions on Calabi-Yau manifolds, presenting in particular the case of the quintic threefold (C. Li's Theorem) with applications and, if time permits, of abelian and hyper-Kähler varieties.

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