A SMOOTH COMPACTIFICATION OF SPACES OF STABILITY CONDITIONS: THE CASE OF THE A_n-QUIVER

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Abstract. We propose a notion of multi-scale stability conditions with the goal of providing a smooth compactification of the quotient of the space of projectivized Bridgeland stability conditions by the group of autoequivalence. For the case of the 3CY category associated with the A_n-quiver this goal is achieved by defining a topology and complex structure that relies on a plumbing construction.

We compare this compactification to the multi-scale compactification of quadratic differentials and briefly indicate why even for the Kronecker quiver this notion needs refinement to provide a full compactification.

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1. Introduction

Spaces of Bridgeland stability on a triangulated category $\mathcal{D}$ have been introduced in [Bri07]. By definition these spaces $\mathrm{Stab}(\mathcal{D})$ are non-compact, in fact they admit a $\mathbb{C}$-action that allows to rescale the central charges. The projectivizations $\mathbb{P}\,\mathrm{Stab}(\mathcal{D}) = \mathbb{C}\setminus \mathrm{Stab}(\mathcal{D})$ are still non-compact, since the ratio of masses of some objects may go to zero. Recently several partial compactifications have been proposed ([BDL20; Bol20; KKO22; BPPW22]), whose merits we compare at the end of the introduction. Our goal is to provide a generalized notion of stability conditions that could provide a *smooth* compactification in the sense of orbifolds of the quotient $\mathbb{C}\setminus \mathrm{Stab}(\mathcal{D})/\mathcal{A}ut(\mathcal{D})$. In this paper we achieve this goal for the CY_3-categories $\mathcal{D}^3_Q$ where Q is a quiver of A_n-type.

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Our approach is motivated by the isomorphism of Bridgeland-Smith [BS15] of $\mathrm{Stab}(\mathcal{D}^3_Q)$ with spaces of quadratic differentials with simple zeros and the generalization of this isomorphism to differentials with higher order zeros constructed in our previous paper [BMQS22]. Its main result states that these are isomorphic to spaces of stability conditions on quotient categories $\mathcal{D}^3_Q/\mathcal{D}^3_{Q_I}$ for some subquivers $Q_I \subset Q$. In both contexts, simple and higher order zeroes, part of the isomorphism is given by identifying central charges of (simple and stable) objects with the distance between zeroes with respect to the metric induced by a quadratic differential. A first and naive idea would be to interpret collision of zeroes of a quadratic differential as the vanishing of central charges. To get a smooth compactification this idea has to be refined.

Our approach is also motivated by the smooth compactification [BCGGM3] of strata of differentials by multi-scale differentials. From there we take the idea that if central charges go to zero we 'zoom in', i.e., we rescale and get another non-zero 'central charge' on a subcategory. This 'central charge' in turn might vanish on some simple objects and forces us to rescale again, thus arriving at a filtration of subcategories. From multi-scale differentials we also borrow the observation that the result of the rescaling process is only well-defined up to multiplication by a common scalar factor, resulting in the definition of equivalence below.

Combining these ideas we can now paraphrase our main notion, see Definition 4.1 for the precise formulation. A *non-split multi-scale stability condition* $(\mathcal{A}_\bullet, Z_\bullet)$ on a triangulated category consists of

- a *multi-scale heart* $\mathcal{A}_\bullet = (\mathcal{A}_i)$, i.e., a collection $\mathcal{A}_L \subset \cdots \mathcal{A}_1 \subset \mathcal{A}_0$ of abelian categories, and
- a *multi-scale central charge*, i.e., a collection $Z_\bullet = (Z_i)_{i=0}^L$ of non-zero $\mathbb{Z}$-linear maps on the Grothendieck groups $Z_i : K(\mathcal{A}_i) \to \mathbb{C}$, where Z_i factors through $\mathrm{Ker}(Z_{i-1})$,

with the following properties. First, the categories $\mathcal{A}_i$ are hearts of the 'vanishing' triangulated subcategories $\mathcal{V}_i^Z \subset \mathcal{D}$ generated by objects $E \in \mathcal{A}_{i-1}$ such that the central charge of the previous filtration step vanishes, i.e. $Z_{i-1}(E) = 0$. Second, the central charges Z_i map simples in $\mathcal{A}_i \setminus \mathcal{A}_{i+1}$ to the semi-closed upper half-plane $\overline{\mathbb{H}}$. (This implies that $\mathcal{V}_{i+1}^Z \cap \mathcal{A}_i$ is a Serre subcategory of $\mathcal{A}_i$.) Third, the induced quotient heart with quotient central charge $(\overline{\mathcal{A}}_i, \overline{Z}_i)$ is a stability condition in the usual sense of [Bri07] on the quotient category $\mathcal{V}_i^Z/\mathcal{V}_{i+1}^Z$. We say that two non-split multi-scale stability conditions are *equivalent* if the induced quotients $(\overline{\mathcal{A}}_i, \overline{Z}_i)$ are projectively equivalent for all $i \geq 1$. We denote by $\mathrm{MStab}(\mathcal{D})$ the set of equivalence classes of those multi-scale stability conditions and add a circle (e.g. $\mathrm{MStab}^\circ(\mathcal{D})$) to denote a specific connected component or a set of reachable stability conditions.

In this paper we only consider multi-scale stability conditions that are *non-split and thus drop this adjective* from now on. In Section 1.2 below we will explain why this notion needs refinements to provide compactifications for more general categories $\mathcal{D}$, even for other CY_3 quiver categories $\mathcal{D}^3_Q$.

We recall that for $\mathcal{D}^3_Q$ of type A_n the group of autoequivalences $\mathcal{A}ut^\circ(\mathcal{D}^3_{A_n})$ preserving a connected component of $\mathrm{Stab}(\mathcal{D}^3_{A_n})$ (modulo those acting trivially) is an extension of $\mathbb{Z}/(n+3)\mathbb{Z}$ by the spherical twist group $\mathrm{ST}(A_n)$, which is isomorphic to a braid group, see Section 3.3.

Theorem 1.1. *The quotient* $\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})/\mathcal{A}ut^\circ(\mathcal{D}^3_{A_n})$ *of the space of multi-scale stability conditions has a structure of a complex orbifold. The projectivization of* *this orbifold* $\mathbb{C}\backslash\,\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})/\mathcal{A}ut^\circ(\mathcal{D}^3_{A_n})$ *is a compactification of the space of projectivized stability conditions up to autoequivalence* $\mathbb{C}\backslash\,\mathrm{Stab}^\circ(\mathcal{D}^3_{A_n})/\mathcal{A}ut^\circ(\mathcal{D}^3_{A_n})$

As a complex orbifold, the space $\mathbb{C}\backslash\,\mathrm{Stab}^\circ(\mathcal{D}^3_{A_n})/\mathcal{A}ut^\circ(\mathcal{D}^3_{A_n})$ is simply the moduli space of curves $\mathcal{M}_{0,n+2}$. The compactification $\mathbb{C}\backslash\,\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})/\mathcal{A}ut^\circ(\mathcal{D}^3_{A_n})$ is however not equal to the Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n+2}$. It is rather a blowup of the latter, as we explain in Section 6.

1.1. **Techniques.** One important technique is the *plumbing* of a multi-scale stability condition, depending on complex numbers τ_i for $i = 0, \ldots, L$, that builds a usual stability condition. If $\tau_i \in i\mathbb{R}_-$ is purely imaginary for all i the result is just the top level heart $\mathcal{A}_0$ together with a central charge that is a rescaled linear combination of the Z_i. One should envision that the size of Z_i is $e^{-\pi i \tau_i}$, thus very small for τ_i close to $-i\infty$ and this is continuously completed by declaring that $\tau_i = -i\infty$ means no plumbing at all. The process of plumbing becomes interesting for τ_i not purely imaginary. This involves rotating $\mathcal{A}_i$. The higher level hearts $\mathcal{A}_{i-1}$ etc. then have to be modified to still contain the rotated heart while still providing the same quotient heart. This modification of the representative however causes that the plumbing action of $(\tau_1, \ldots, \tau_L) \in -\mathbb{H}^L$ is not the action of a semigroup: the semigroup addition and the action only almost commute, with an error that goes to zero as $\tau_i \to -i\infty$.

In this way, we give $\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})$ a topology by declaring neighborhoods of a multi-scale stability conditions to be plumbings with $t_i := e^{-\pi i \tau_i}$ small composed with a small deformation of the stability condition. However, this space is not locally compact. In fact, for $n = 2$ the space is isomorphic to $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ with the horoball topology, as we will explain in Section 6.4.

The complex orbifold structure on the quotient $\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})/\mathcal{A}ut^\circ(\mathcal{D}^3_{A_n})$ is locally given by the functions t_i together with the central charges Z_i. This statement requires to control the stabilizer of a neighborhood of the multi-scale stability condition. We show that this stabilizer contains a finite index subgroup isomorphic to $\mathbb{Z}^L$.

For compactness of $\mathbb{C}\backslash\,\mathrm{Stab}^\circ(\mathcal{D}^3_{A_n})/\mathcal{A}ut^\circ(\mathcal{D}^3_{A_n})$ the obvious idea is to normalize in a given sequence of multi-scale stability conditions the mass of the largest simple to be one, and then define an order on the set of simples corresponding to the speed in which their central charges go to zero. The level sets for this order will then correspond to the index set of the limiting multi-scale stability condition. The challenge for this idea arises, if the central charge of a stable but non-simple object tends to zero despite the normalization while the central charge of its simple factors do not. This forces the central charge of some simple object to tend to the positive real axis.

Consider for example the sequence $\sigma_n = (\mathcal{A}, Z_n)$ of stability conditions on $\mathcal{D}^3_{A_2}$, all supported on a fixed heart $\mathcal{A}$ and with

$$Z_n(S_1) = -1 + i/n, \qquad Z_n(S_2) = 1 + i/n\,. \tag{1}$$

see Figure 1 for the picture of the corresponding quadratic differential. In the limit $n \to \infty$, the central charge vanishes precisely on the subcategory generated by the non-trivial extension E of S_1 by S_2. Since E is not simple, it does not define a non-trivial Serre subcategory of $\mathcal{A}$, contradicting a consequence of our definition of multi-scale stability condition.

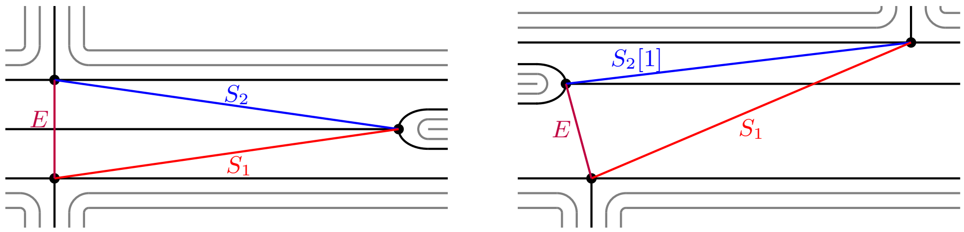

FIGURE 1. Quadratic differential illustrating a degenerating sequence in $\mathrm{Stab}^\circ(\mathcal{D}^3_{A_2})$ and a rotated situation

The solution to find the limiting object is to rotate the sequence by λ_n so that $Z_{\lambda_n\sigma_n}(S_2) \in \mathbb{H}^-$, see Figure 1 on the right. The heart $\mathcal{A}$ is replaced by $\mathcal{A}_0 := \mu_{S_2}\mathcal{A}$, the tilt one would usually perform also inside $\mathrm{Stab}(\mathcal{D}^3_{A_2})$ if the central charge of the simple S_2 approaches the positive real axis. Now E is simple and the vanishing category $\mathcal{V}_1^Z$ generated by E has the property that $\mathcal{V}_1^Z \cap \mathcal{A}_0$ is Serre in $\mathcal{A}_0$. The limiting multi-scale stability condition consists of the filtration $\mathcal{A}_0 \supset \mathcal{A}_1 = \langle E\rangle$ together with $Z_0(S_2[1]) = -1$ and $Z_0(E) = 0$ as well as $Z_1(E)$ arbitrary non-zero in view of the notion of equivalence.

1.2. **Obstructions to generalization.** Continuing the idea of proof for compactness we consider one of the simplest cases beyond A_n-type quivers, stability conditions on the CY_3-category of the Kronecker quiver, or, in the language of quadratic differentials (see [BS15, Example 12.5]), the stratum $\mathcal{Q}(-3,-3,1,1)$ with two triple poles and two simple zeros. We again consider a situation where the central charge of a stable but non-simple object tends to zero within the normalization that the mass of the largest simple is approximately one. Now using a small rotation does not seem to help. The compactification of strata of differentials ([BCGGM3; BCGGM2]) that we recall in Section 6 hints to the reason for this problem.

The boundary strata of the compactification are encoded by level graphs, whose vertices correspond to components of stable curves and where a vertex v_1 is above a vertex v_2 if the differential tends to zero on v_2 more quickly than on v_1. In terms of multi-scale stability conditions we find the same level structure (given by index of $\mathcal{A}_i$) and components (given by the components of the ext-quiver on the simples in $\mathcal{A}_i$). However, for differentials we allow for *horizontal degenerations*, i.e., edges between vertices on the same level. In this degeneration of the Kronecker quiver alluded to above (with central charge as in (1)) we should normalize the sequence to keep the length of the 'short' stable object E (i.e. the extension of S_1 by S_2 or geometrically the length of the core curve of the cylinder) constant. This happens at the expense of letting the mass of both simples go to infinity. In the geometric picture, the surface splits into two subsurfaces with quadratic differentials of type $(-3,-2,1)$. It would be interesting to enlarge the concept of non-split multi-scale differentials so as to include this 'splitting' of the category.

It seems quite plausible that the current definition of (non-split) multi-scale stability condition provides a partial compactification of $\mathbb{C}\backslash\,\mathrm{Stab}^\circ(\mathcal{D}^3_Q)/\mathcal{A}ut^\circ(\mathcal{D}^3_Q)$ to a complex orbifold for general quiver categories (or whenever $\mathrm{Stab}^\circ(\mathcal{D})$ is of tame type). This requires to overcome several technical problems that we highlight along with the definition of the topology in Section 4. Currently we rely on the fact that hearts in $\mathcal{D}^3_{A_n}$ have finitely many indecomposables.

1.3. **Comparison to other compactifications.** We are aware of four other papers aiming to compactify spaces of stability conditions. Bolognese [Bol20] uses a metric completion to give a partial compactification. The Thurston-type compactifications of Bapat, Deopukar and Licata [BDL20] and Kikuta-Koseki-Ouchi [KKO22] use the tuple of all masses to get a map from the space of projectivized stability conditions to some projective space and take the closure there. The space of lax stability conditions of Broomhead, Pauksztello, Plog, Woolf [BPPW22] allows some of the masses of semistable objects to be zero, but requires a modified support property and zero being an isolated point of the set of all masses.

Common to all these approaches is that they aim to (partially) compactify the space $\mathbb{P}\,\mathrm{Stab}(\mathcal{D})$ of projectivized stability conditions whereas we compactify its quotient by $\mathcal{A}ut^{\circ}(\mathcal{D}^3_{A_n})$. Moreover in all these four papers the boundary or boundary strata are real codimension one, whereas in our approach the boundary has complex codimension one, since we construct a complex orbifold. We mention that [BCGGM3, Section 15] proposes a real-oriented blowup of the complex orbifold, thus a real manifold with corners, to which the $\mathrm{GL}_2^+(\mathbb{R})$-action extends. This real-oriented blowup construction can certainly also be incorporated into a modified definition of multi-scale stability conditions. In this real blowup there are real codimension one boundary strata, which parametrize stability conditions on a quotient category by a rank one subcategory together – this is effect of the real blowup – with the phase of the simple with vanishing mass. This seems to agree with the codimension one boundary strata of [BPPW22]. Since both approaches, ours and [BPPW22], use stability conditions on quotient categories and the difficulties often stem from lifting problems, it would be interesting to compare or combine them. However this does not seem to solve the problem of getting a compact space in a more general setting, as we see no subsitute for the missing 'horizontal degenerations'.

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2. Background and notation

In this section we recall basic material about stability conditions, quivers and their CY_3-categories $\mathcal{D}^3_Q$. References for this includes [HRS96; GM03; BBD82; Bri07; Bri09; Nee14; DWZ08; Kel11].

2.1. **Notation and fundamental assumptions.** We fix some notation that will be used throughout. Let k denote an algebraically closed field, and any category is additive, k-linear, and essentially small. We deal with finite-dimensional abelian and triangulated categories of modules (resp., dg modules) over a finite-dimensional algebra (resp., a dg algebra). Whenever we define a subcategory, we mean that there is a fully faithful functor that we assume to be the embedding.

Given subcategories $\mathcal{A}_1, \mathcal{A}_2$ of an abelian or a triangulated category $\mathcal{C}$, and a set of objects $\mathcal{B}$, we define (usually omitting the subscript $\mathcal{C}$)

$$\mathcal{A}_1 \perp_{\mathcal{C}} \mathcal{A}_2 := \{M \in \mathcal{C} \mid \exists \text{ s.e.s or triangle } T \to M \to F \text{ s.t. } T \in \mathcal{A}_1, F \in \mathcal{A}_2\},$$
$$\text{if } \mathrm{Hom}(H_1, H_2) = 0 \text{ for any } H_1 \in \mathcal{A}_1\,, H_2 \in \mathcal{A}_2,$$

We define $\langle \mathcal{B} \rangle$ depending on the context to be the *abelian category generated by* $\mathcal{B}$, the thick *triangulated category generated by* $\mathcal{B}$, the *torsion-free class* or the *torsion class generated by* $\mathcal{B}$.

Let $\mathcal{A}$ be an abelian category. It is called a (finite) length category if any object $E \in \mathcal{A}$ admits a finite sequence of subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$$

such that all E_i/E_{i-1} are simple. It is called finite if moreover it has finitely many simple objects. If an abelian category is finite, its Grothendieck group is generated by the isomorphism classes of its simples.

Let $\mathcal{D}$ be a triangulated category. For simplicity we make the strong assumption that its Grothendieck group is a finite rank lattice $K(\mathcal{D}) \simeq \mathbb{Z}^{\oplus n}$. This is not the general situation, though it will hold for the most relevant categories considered later. The main reason for such an hypothesis is to simplify the definition of a (multi-scale) stability condition.

Definition 2.1. *A* bounded t-structure *on a triangulated category* $\mathcal{D}$ *is the datum of a full additive subcategory* $\mathcal{P} \subset \mathcal{D}$ *stable under positive shift such that* $\mathcal{P} \perp \mathcal{P}^{\perp} = \mathcal{D}$*, and moreover* $\mathcal{D}$ *is generated by* $\cup_{m \in \mathbb{Z}} \big(\mathcal{P}[m] \cap \mathcal{P}^{\perp}[-m]\big)$. *The* heart *of a bounded t-structure is the subcategory* $\mathcal{P} \cap \mathcal{P}^{\perp}[1]$.

The heart $\mathcal{A}$ of a bounded t-structure is an abelian category. The cohomological functor $H^0 : \mathcal{D} \to \mathcal{A}$ realizes an isomorphism at the level of Grothendieck groups

$$H^0_* : K(\mathcal{A}) \simeq K(\mathcal{D}).$$

Moreover, a bounded t-structure is uniquely determined by its heart as $\mathcal{P} = \langle \mathcal{H}[i], i \geq 0 \rangle$. For this reason we will speak about a t-structure or its heart interchangeably.

There is a partial order on hearts $\mathcal{A}_1 \leq \mathcal{A}_2$ defined by $\mathcal{P}_1 \supset \mathcal{P}_2$ or equivalently $\mathcal{P}_1^{\perp} \supset \mathcal{P}_2^{\perp}$. A heart $\mathcal{H}$ will be called *intermediate* with respect to a fixed heart $\mathcal{A}$ if $\mathcal{A} \leq \mathcal{H} \leq \mathcal{A}[1]$.

2.2. **Torsion pairs and tilting.** A *torsion pair* for an abelian category $\mathcal{A}$ consists on a pair $(\mathcal{T}, \mathcal{F})$ of full additive subcategories of $\mathcal{A}$ called *torsion class* and *torsion-free class*, such that $\mathcal{A} = \mathcal{T} \perp \mathcal{F}$. In other words, a torsion pair mimics a bounded t-structure at abelian level. In fact, a torsion pair in the heart of a bounded t-structure $\mathcal{A}$ in a $\mathcal{D}$ defines new bounded t-structures with hearts

$$\mu^{\sharp}_{\mathcal{F}}\mathcal{A} := \mathcal{T} \perp_{\mathcal{D}} \mathcal{F}[1], \quad \mu^{\flat}_{\mathcal{T}}\mathcal{A} := \mathcal{F} \perp_{\mathcal{D}} \mathcal{T}[-1]$$

They are called respectively the *forward tilt* at $\mathcal{F}$ (resp. *backward tilt*) at $\mathcal{T}$, [HRS96]. They are related by $\mu^{\sharp}_{\mathcal{T}[-1]}\mu^{\flat}_{\mathcal{T}}\mathcal{A} = \mathcal{A}$ and $\mu^{\flat}_{\mathcal{F}[1]}\mu^{\sharp}_{\mathcal{F}}\mathcal{A} = \mathcal{A}$. The forward tilt of $\mathcal{A}$ at a torsion-free class is intermediate with respect to $\mathcal{A}$; the backward tilt of $\mathcal{A}$ at a torsion class is intermediate with respect to $\mathcal{A}[-1]$.

In a finite abelian category torsion and torsion free classes are closed under extensions and are characterized by being closed under quotients and subobjects respectively. This implies that any Serre subcategory is both torsion and torsion-free class. When we tilt at a torsion(-free) class $\langle S \rangle$ generated by a simple object S, we speak about a simple tilt and we simplify the notation to $\mu^{\sharp}_{S}\mathcal{A}$ and $\mu^{\flat}_{S}\mathcal{A}$. Suppose $\mathcal{A}$ is a finite heart with simple objects $\mathrm{Sim}(\mathcal{A}) := \{S_1, \ldots, S_n\}$, which are rigid, i.e., have no non-trivial self-extensions, and let $S \in \mathrm{Sim}(\mathcal{A})$. Then

$$\mathrm{Sim}\,\mu^\sharp\mathcal{A} = \{S[1]\} \cup \{\mathrm{Cone}\left(S \xrightarrow{ev} S[1] \otimes \mathrm{Ext}^1(T,S)^*\right)[-1],\ S \neq T \in \mathrm{Sim}\,\mathcal{A}\}$$
$$\mathrm{Sim}\,\mu^\flat\mathcal{A} = \{S[-1]\} \cup \{\mathrm{Cone}\left(S[-1] \otimes \mathrm{Ext}^1(S,T) \xrightarrow{ev} T\right),\ S \neq T \in \mathrm{Sim}\,\mathcal{A}\},$$

see e.g. [KQ15]. Note also that the simple tilting of a finite heart in $\mathcal{D}$ is another finite heart. In some cases tilting at a torsion (or torsion-free) class can be decomposed into a finite sequence of simple tilts.

Proposition 2.2. *Suppose that $\mathcal{A}$ is a finite heart.*

(1) *Tilting at torsion theory in $\mathcal{A}$ containing only finitely many indecomposables is equivalent to performing a sequence of simple forward tilts.*
(2) *Conversely, suppose $a_1, \dots, a_k$ is a finite sequence of objects in $\mathcal{A}$ such that $a_i \in \mathcal{A}$ is simple in $\mu^\sharp_{a_{i-1}} \dots \mu^\sharp_{a_1}\mathcal{A}$. Then $\mu^\sharp_{a_k} \dots \mu^\sharp_{a_1}\mathcal{A} = \mu^\sharp_{\mathcal{F}}\mathcal{A}$ where $\mathcal{F} = \langle a_0, \dots, a_k\rangle$.*
(3) *More generally, for any two torsion-free classes $\mathcal{F}_1 \subset \mathcal{F}_2$ with $\mathcal{F}_2$ having finitely many indecomposables, there is a sequence of simple tilts at objects a_i such that $\mu^\sharp_{\mathcal{F}_2} = \mu^\sharp_{a_k} \cdots \mu^\sharp_{a_1}\mu^\sharp_{\mathcal{F}_1}$ and $\mathcal{F}_2 = \langle \mathcal{F}_1, a_1, \cdots, a_k\rangle$.*

Proof. For the proof of the first two items see [Woo10, Proof of Proposition 2.4] and use the relation $\mu^\sharp_{\mathcal{T}[-1]}(\mu^\flat_{\mathcal{T}}\mathcal{A}) = \mathcal{A}$ to convert the statement about backward tilts in loc. cit. to the given version. The last statement follows from [HLŠV22, Section 7.1] (see in particular Proposition 7.5). □

2.3. **Bridgeland stability conditions.** Recall from [Bri07] that a *stability condition* σ on a triangulated category $\mathcal{D}$ is a pair $\sigma = (\mathcal{A}, Z)$, consisting of the heart of a bounded t-structure $\mathcal{A}$, together with a central charge $Z \in \mathrm{Hom}(K(\mathcal{A}), \mathbb{C})$, i.e., a group homomorphism that maps the class of non-zero elements in $\mathcal{A}$ to the semi-closed half plane $\overline{\mathbb{H}} := \{re^{\pi i\theta} \in \mathbb{R} | r \in \mathbb{R}_{>0},\ 0 < \theta \le 1\}$ and that satisfies the support property and Harder-Narasimhan condition of loc. cit. We fix a finite rank lattice K and a surjective morphism $\nu : K(\mathcal{A}) \to K$ and require that Z factors through ν. In the case $K(\mathcal{A}) \simeq \mathbb{Z}^n$ we require that $K = K(\mathcal{A})$ and $\nu = \mathrm{id}$.

We use that stability conditions can equivalently be specified as a $\sigma = (\mathcal{P}, Z)$ using a central charge and a slicing, compatible in the sense that $E \in \mathcal{P}(\phi)$ implies $Z([E]) = m\exp(\pi i\phi)$ for some positive $m \in \mathbb{R}$.

An object $E \in \mathcal{D}$ is called *σ-semistable* if $E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$. It is called *σ-stable* if it is simple in $\mathcal{P}(\phi)$. This notion makes sense because any subcategory $\mathcal{P}(\phi)$ is abelian if $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi\in\mathbb{R}}$ is a slicing compatible with $Z \in \mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$.

Let $\lambda \in \mathbb{C}$ and suppose $0 < \epsilon = \mathrm{Re}(\lambda) \le 1$. We observe, and will use later, that σ-semistable objects X in the heart $\mathcal{A}$ (equivalently Z-semistable objects) with $1 - \epsilon \le \phi(X) < (\le)1$ and those with $0 < \phi(X) \le (<)1 - \epsilon$, for $\epsilon \in (0, 1)$, form a torsion pair $(\mathcal{T}_\lambda, \mathcal{F}_\lambda)$ in $\mathcal{A}$ due to the Harder-Narasimhan condition.

The space of stability conditions is a complex manifold $\mathrm{Stab}(\mathcal{D})$. There are two natural commuting actions:

- a left action by $\mathbb{C}$, by rescaling the central charge and tilting the heart, if $0 < \mathrm{Re}(\lambda) \le 1$,
$$\lambda \cdot (\mathcal{A}, Z) \ = \ (\mu_{\mathcal{F}_\lambda}\mathcal{A}, e^{-\pi i\lambda}Z),$$
- and a right action by $\mathrm{Aut}(\mathcal{D})$ via pullback,
$$\Phi.(\mathcal{A}, Z) \ = \ \left(\Phi\mathcal{A}, Z \circ [\Phi]^{-1}\right),$$

where $[\Phi]$ is the map induced by Φ on $K(\mathcal{D})$.

In particular the shift [1] acts as $\lambda = 1$. Note that the $\mathbb{C}$-action does not change the notion of semistability and stability.

We denote by $\mathrm{Stab}^\circ(\mathcal{D})$ be a connected component, specified by the context. The stability manifold $\mathrm{Stab}(\mathcal{D})$ is tiled into subsets $\mathrm{Stab}(\mathcal{A})$ of stability conditions supported on the heart $\mathcal{A}$. The component $\mathrm{Stab}^\circ(\mathcal{D})$ is called *finite type* if is the union of $\mathrm{Stab}(\mathcal{A})$ over *finite hearts*. It is called of *tame type*, if the $\mathbb{C}$-orbits of $\mathrm{Stab}(\mathcal{A})$ for all finite type hearts cover $\mathrm{Stab}^\circ(\mathcal{D})$.

We let $\mathrm{Aut}^\circ(\mathcal{D})$ be the subgroup of $\mathrm{Aut}(\mathcal{D})$ consisting on autoequivalences of $\mathcal{D}$ that preserve the component $\mathrm{Stab}^\circ(\mathcal{D})$ and we define $\mathrm{Nil}^\circ(\mathcal{D}) \subset \mathrm{Aut}^\circ(\mathcal{D})$ the subgroup of *negligible autoequivalences*, i.e. those that act trivially on $\mathrm{Stab}^\circ(\mathcal{D})$. We use fancy fonts like

$$\mathcal{A}ut^\circ(\mathcal{D}) \;=\; \mathrm{Aut}^\circ(\mathcal{D})/\,\mathrm{Nil}^\circ(\mathcal{D}) \tag{2}$$

to denote the quotient groups by negligible autoequivalences. It's the quotient spaces $\mathbb{C}\backslash\,\mathrm{Stab}(\mathcal{D})/\mathcal{A}ut(\mathcal{D})$ by these actions that we want to compactify.

2.4. **Quivers with potential, module and Ginzburg categories.** In this paper (Q, W) is a quiver $Q = (Q_0, Q_1, s, t)$ with potential W (i.e., a formal sum of cycles) up to right-equivalence, see [DWZ08; KY11] for standard results. We assume that (Q, W) has no loops and no 2-cycles, that the set of vertices Q_0 and the set of arrows Q_1 are finite, and that the potential defines a bilateral ideal $\partial W = \langle \partial_a W \mid a \in Q_1\rangle \subset kQ$ such that the Jacobian algebra

$$\mathcal{J}(Q, W) := \widehat{kQ}/\partial W,$$

obtained by quotienting the completed path algebra by the ideal defined by the potential, is finite dimensional. Note that in our case of interest, $\widehat{kQ}/\partial W = kQ/\partial W$. For a ring $\mathcal{J}$, we denote by $\mathrm{Mod}\,\mathcal{J}$ the abelian category of left modules and by $\mathrm{mod}\,\mathcal{J}$ the abelian category of finitely generated left modules. The category $\mathrm{mod}\,\mathcal{J}(Q, W)$ is finite with simple objects $\mathrm{Sim}(\mathrm{mod}\,\mathcal{J}(Q, W)) = \{S_1, \dots, S_n\}$, where $n = |Q_0|$.

If $I \subset Q_0$ is a collection of vertices of (Q, W), by (Q_I, W_I) we mean the *restriction* of (Q, W) to I. It is another finite quiver with potential, possibly disconnected, defined by $(Q_I)_0 = I$, $(Q_I)_1 = \{a : i \to j \in Q_1 \mid i, j \in I\}$, and with source, tail functions, and potential obtained by restriction from (Q, W) to I. We call it a (full) subquiver. The complement of I in Q_0 will be denoted I^c.

The *mutation* of a quiver with potential (Q, W) at a vertex i is an operation that produces another quiver with the same set of vertices and new set of arrow and new potential, defined as follows. From Q_1, keep all arrow not incident to i; replace any arrow a with either $s(a)$ or $t(a)$ equal to i with its opposite; add an arrow $[ab] : k_1 \to k_2$ for any pair of consecutive arrow $a : k_1 \to i$ and $b : i \to k_2$; finally, remove any two-cycles. The new potential is the formal sum of W and $\sum_{a,b\in Q_1} [ab]b^*a^*$.

The *Ginzburg algebra* of (Q, W) is a dg algebra denoted $\Gamma(Q, W)$ and introduced in [Gin06; KY11]. It does not depend on the mutation class of a quiver with potential.

Definition 2.3. *The* perfectly-valued derived category $\mathrm{pvd}(\Gamma)$ *associated with a dg algebra* Γ *is the subcategory of the derived category* $\mathcal{D}(\Gamma)$ *consisting on dg modules with finite dimensional total cohomology.*

Once we fixed (Q, W) and $I \subset Q_0$, we write $\mathcal{J} = \mathcal{J}(Q, W)$, $\Gamma = \Gamma(Q, W)$, and $\mathcal{J}_I = \mathcal{J}(Q_I, W_I)$, $\Gamma_I = \Gamma(Q_I, W_I)$. We have the following inclusion of triangulated categories, [KY11]

$$\mathrm{pvd}(\Gamma) \subset \mathrm{per}(\Gamma) \subset \mathcal{D}(\Gamma).$$

It is proven in [KY11] that the *standard* t-structure with heart $\mathrm{Mod}\,\mathcal{J}$ in the derived category $\mathcal{D}(\Gamma)$ restricts to $\mathrm{per}(\Gamma)$ and $\mathrm{pvd}(\Gamma)$, on which it defines a bounded t-structure with heart $\mathrm{mod}\,\mathcal{J}$, that we call *standard* as well.

The perfectly-valued derived category of the Ginzburg algebra of a quiver with potential is 3-Calabi-Yau, which means that for any objects $E, F \in \mathrm{pvd}(\Gamma)$ there is a natural isomorphism of k-vector spaces $\nu : \mathrm{Hom}(E, F) \xrightarrow{\sim} \mathrm{Hom}(F, E[3])^\vee$. Moreover, the simple objects in the standard heart $\mathrm{mod}\,\mathcal{J}$ are *spherical* in $\mathrm{pvd}(\Gamma)$, see [Kel11, Lemma 4.4] and [KQ15, Corollary 8.5].

If two quivers with potential (Q, W) and (Q', W') are related by mutations, then $\mathcal{D}(\Gamma(Q, W)) \simeq \mathcal{D}(\Gamma(Q', W'))$ and $\mathrm{pvd}(\Gamma(Q, W)) \simeq \mathrm{pvd}(\Gamma(Q', W'))$. Therefore $\mathrm{mod}\,\mathcal{J}(Q', W')$ is viewed as another heart of bounded t-structure of $\mathrm{pvd}\,\Gamma(Q, W)$. We recall that in general not all bounded t-structures have this shape.

It is clear that any property of $\mathrm{pvd}(\Gamma)$ and $\mathrm{mod}(\mathcal{J})$ also holds for $\mathrm{pvd}(\Gamma_I)$ and $\mathrm{mod}\,\mathcal{J}_I$.

As explained in [KY18], the Ginzburg algebra Γ_I is isomorphic to $\Gamma/\Gamma e\Gamma$, where $e = \sum_{i \in I^c} e_i$ is the idempotent in Γ associated to the complement $I^c = Q_0 \setminus I$. On the other hand the dg algebra $e\Gamma e$ is the endomorphism algebra of the projective module $\Gamma e = \sum_{i \in I^c} \Gamma e_i$ in $\mathcal{D}(\Gamma)$ and the Verdier quotient $\mathcal{D}(\Gamma)/\mathcal{D}(\Gamma_I)$ coincides with $\mathcal{D}(e\Gamma e)$. Similarly $\mathcal{J}_I = \mathcal{J}/\mathcal{J}e\mathcal{J}$ and the quotient perfectly valued and abelian categories that will be relevant in the rest of the paper are $\mathrm{pvd}\, e\Gamma e$ and $\mathrm{mod}\, e\mathcal{J}e$:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{pvd}\,\Gamma_I \arrow[r] \arrow[d, "H^0"] & \mathrm{pvd}\,\Gamma \arrow[r] \arrow[d, "H^0"] & \mathrm{pvd}\, e\Gamma e \arrow[r] \arrow[d, "H^0"] & 0 \\ 0 \arrow[r] & \mathrm{mod}\,\mathcal{J}_I \arrow[r] & \mathrm{mod}\,\mathcal{J} \arrow[r] & \mathrm{mod}\, e\mathcal{J}e \arrow[r] & 0. \end{tikzcd}$$

The last line is part of a recollement of abelian categories, described for instance in [Psa18].

In the rest of the paper we let

- $\mathcal{D}^3_Q$ be the 3-Calabi-Yau triangulated category $\mathrm{pvd}\,\Gamma(Q, W)$.

The case of primary interest will be quivers *of type* A_n, i.e., that can be obtained with by finite sequence of mutations from the quiver

$$A_n := \bullet_1 \longrightarrow \bullet_2 \longrightarrow \cdots \longrightarrow \bullet_n\ , \quad n \geq 1.$$

Any restriction of a quiver of type A_n is a union of quivers of type A_m's.

Given an A_n-configuration, and the abelian category $\mathrm{mod}\,\mathcal{J}(A_n)$, we denote by S_i the simple module associated with the vertex i. For $i \leq k$, we denote by $S_{i\ldots k}$ the $\mathcal{J}(A_n)$-module defined inductively as the indecomposable fitting into the short exact sequence

$$0 \to S_{k+1} \to S_{i\ldots(k+1)} \to S_{i\ldots k} \to 0. \tag{3}$$

The $S_{i\ldots k}$, are the projective resolutions P_i of S_i in the abelian subcategory $\langle S_i, \ldots, S_k \rangle$ which is of $A_{(k-i)}$-type by construction.

3. Stability manifolds for marked surfaces

Decorated marked surfaces are one of the natural sources for quiver categories. They are well-studied thanks also to the Bridgeland-Smith isomomorphism [BS15] to spaces of quadratic differentials with simple zeros. We recall this result here, together with the generalization in our previous paper [BMQS22]. This setup contains our main case study, the A_n-quiver, and serves as motivation for the use of quotient categories. Triangulations of decorated marked surfaces will serve as reference point to pick out the right connected components of stability manifolds needed in the later sections.

3.1. **The stability manifold of a decorated marked surface.** A natural way to construct quivers is from triangulations of surfaces and we will use this formalism to keep track of connected components of stability spaces and later the multi-scale stabilty conditions.

A *marked surface* $\mathbf{S} = (\mathbf{S}, \mathbb{M}, \mathbb{P})$ consists of a connected bordered differentiable surface with a fixed orientation, with a finite set $\mathbb{M} = \{M_i\}_{i=1}^b$ of marked points on the boundary $\partial\mathbf{S} = \bigcup_{i=1}^b \partial_i$, and a finite set $\mathbb{P} = \{p_j\}_{j=1}^p$ of punctures in its interior $\mathbf{S}^\circ = \mathbf{S} - \partial\mathbf{S}$, such that each connected component of $\partial\mathbf{S}$ contains at least one marked point.

A *decorated marked surface* $\mathbf{S}_\Delta$ (abbreviated as *DMS*) is obtained from a marked surface $\mathbf{S}$ by decorating it with a set $\Delta = \{z_i\}_{i=1}^r$ of points in the surface interior $\mathbf{S}^\circ$. These points are called *finite critical points* or *finite singularities*.

An *open arc* is an (isotopty class of) curve $\gamma\colon I \to \mathbf{S}_\Delta$ such that its interior is in $\mathbf{S}_\Delta^\circ \setminus \Delta$ and its endpoints are in the set of marked points $\mathbb{M}$. An *(open) arc system* $\{\gamma_i\}$ is a collection of open arcs on $\mathbf{S}_\Delta$ such that there is no (self-)intersection between any of them in $\mathbf{S}_\Delta^\circ \setminus \Delta$. A *triangulation* $\mathbb{T}$ of $\mathbf{S}_\Delta$ is a maximal arc system of open arcs, which in fact divide $\mathbf{S}_\Delta$ into triangles.

The quiver $Q_\mathbb{T}$ with potential $W_\mathbb{T}$ associated to a triangulation $\mathbb{T}$ is constructed as follows. The vertices correspond to the open arcs in $\mathbb{T}$, the arrows of $Q_\mathbb{T}$ correspond to oriented intersection between open arcs in $\mathbb{T}$, so that there is a 3-cycle in $Q_\mathbb{T}$ locally in each triangle, and the potential $W_\mathbb{T}$ is the sum of all such 3-cycles.

For a fixed initial triangulation $\mathbb{T}_0$ we denote by $\Gamma_{\mathbb{T}_0} = \Gamma(Q_{\mathbb{T}_0}, W_{\mathbb{T}_0})$ the Ginzburg algebra associated with the quiver associated with $\mathbb{T}_0$ we let $\mathcal{D}^3_{Q_{\mathbb{T}_0}} = \mathrm{pvd}(\Gamma_{\mathbb{T}_0})$ or simply $\mathcal{D}^3_Q$ the corresponding CY_3-category. Finally, we define $\mathrm{Stab}^\circ(\mathcal{D}^3_Q)$ to be the connected component of the space of Bridgeland stability conditions on $\mathcal{D}^3_Q$ containing stability conditions supported on the standard heart $\mathcal{H}_0$ of $Q_{\mathbb{T}_0}$.

In this paper we fix throughout a DMS $\mathbf{S}$ of type A_n. It is a disc with $b = 1$ boundary component, which has $n + 3$ marked points, $r = n + 1$ finite critical points in its interior, and no punctures. We use this reference surface and a reference triangulation on it to define the component $\mathrm{Stab}^\circ(\mathcal{D}_{A_n})$. Recall from [BS15, Theorem 9.9 and Section 12.1] that the subgroup $\mathrm{Aut}^\circ(\mathcal{D}_{A_n}) \subset \mathrm{Aut}(\mathcal{D}_{A_n})$ preserving a connected component of $\mathrm{Stab}(\mathcal{D}_{A_n})$ is an extension of $\mathbb{Z}/(n+3)\mathbb{Z}$ by the spherical twist group $\mathrm{ST}(A_n)$.

In this language the main theorem of Bridgeland-Smith (for a general CY_3-quiver category $\mathcal{D}^3_Q$ associated with a triangulation of $\mathbf{S}_\Delta$, see [BS15] for the excluded cases) reads:

Theorem 3.1 ([BS15; KQ20])**.** *There is an isomorphism of complex manifolds*

$$K : \mathrm{FQuad}^{\circ}(\mathbf{S}_{\Delta}) \to \mathrm{Stab}^{\circ}(\mathcal{D}^3_Q)\,. \tag{4}$$

This map K is equivariant with respect to the action of the mapping class group $\mathrm{MCG}(\mathbf{S}_{\Delta})$ *on the domain and of the automorphism group $\mathcal{A}ut^{\circ}(\mathcal{D})$ on the range. These groups act properly discontinuously on domain, resp. range.*

Here $\mathrm{FQuad}^{\circ}(\mathbf{S}_{\Delta})$ is a space of framed quadratic differentials with simple zeros at Δ, whose definition we recall along with the examples in Section 6. Its generalization to non-simple zeros motivates the notion of collapse and the use of quotient categories, which we recall in Section 3.2.

As technical tool we introduce the *exchange graph* $\mathrm{EG}(\mathbf{S}_{\Delta})$, the directed graph whose vertices are the triangulations of $\mathbf{S}_{\Delta}$ and whose edges are given by (forward) flips of the triangulation. The *exchange graph* $\mathrm{EG}(\mathcal{D})$ of a triangulated category is the directed graph whose vertices are the finite hearts and whose edges are give by forward tilts at simples in the heart. We denote by $\mathrm{EG}^{\circ}(\mathbf{S}_{\Delta})$ the connected component containing the initial triangulation $\mathbb{T}_0$ and by $\mathrm{EG}^{\circ}(\mathcal{D}^3_Q)$ the connected component corresponding to the standard heart mod $\mathcal{J}(Q_{\mathbb{T}_0}, W_{\mathbb{T}_0})$. A key step in the proof of Theorem 3.1 is the isomorphism

$$\mathrm{EG}^{\circ}(\mathbf{S}_{\Delta}) \;\cong\; \mathrm{EG}^{\circ}(\mathcal{D}^3_Q) \tag{5}$$

of exchange graphs.

3.2. **Stability manifolds of certain quotient categories.** Higher order zeros are modeled by the collapse of a subsurface $\Sigma \subset \mathbf{S}_{\Delta}$ in a DMS. We use this to deduce information on certain components of the stability manifold of the quotient categories $\mathcal{D}^3_Q/\mathcal{D}^3_{Q_I}$. We decompose Σ into connected components Σ_i, provide each boundary component of Σ_i with an integer *enhancement* κ_{ij}. To match hypothesis with [BMQS22] we suppose throughout that $\kappa_{ij} \geq 3$ and consider Σ as a marked surface with κ_{ij} points on each boundary component.

To topologically formalize the collapse of Σ we define a *weighted DMS* (wDMS for short) to be a DMS with a weight function $\mathbf{w} : \Delta \to \mathbb{Z}_{\geq -1}$ where the total weight is required to be $||\mathbf{w}|| = 4g - 4 + |\mathbb{M}| + 2b$. Contracting $\Sigma \subset \mathbf{S}_{\Delta}$ and replacing each boundary component by a decoration point in Δ with weight $\mathbf{w}_{ij} = \kappa_{ij} - 2$ defines a wDMS that we usually denote by $\overline{\mathbf{S}}_{\mathbf{w}}$. In the sequel (as in [BMQS22]) we *restrict to the case of no punctures $p = 0$, no unmarked boundary components and* $\mathbf{w} : \Delta \to \mathbb{Z}_{\geq 1}$.

To categorify the collapse we homotope the initial triangulation $\mathbb{T}_0$ such that the arcs intersect the boundary of Σ in the marked points and such that $\mathbb{T}_0|_{\Sigma}$ is a triangulation of this subsurface. In this way Σ becomes a DMS with a triangulation and we may form the CY_3-category $\mathcal{D}_3(\Sigma)$. We define the Verdier quotient category

$$\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}) \;:=\; \mathcal{D}_3(\mathbf{S}_{\Delta})/\mathcal{D}_3(\Sigma) \tag{6}$$

As in Section 3.1 there are two exchange graphs associated with this situation, one based on "flips" and topology and the other based on tilts of hearts. The isomorphism (7) below between these graphs is one of reasons to work with the quotient categories.

A *partial triangulation* $\mathbb{A}$ *of* $\overline{\mathbf{S}}_{\mathbf{w}}$ is a collection of open arcs that triangulates the subsurface of $\mathbf{S}_{\Delta}$ whose complement is homeomorphic to Σ, and such that each boundary component c_{ij} of Σ is homeomorphic in $\overline{\mathbf{S}}_{\mathbf{w}} \setminus \mathbb{A}$ to a ($\kappa_{ij} = w_{ij} + 2$)-gon, possibly with ends points identified.

On the set of partial triangulation $\mathbb{A}$ there is an operation of *forward flip of an arc* $\gamma \in \mathbb{A}$, defined by moving both endpoints counterclockwise one edge bounding the subsurface of $\mathbf{S}_\Delta \setminus (\mathbb{A} \setminus \{\gamma\})$ that contains γ. This generalizes the usual notion of flip of triangulations, see [BMQS22, Figure 2]. We define the exchange graph $\mathrm{EG}(\overline{\mathbf{S}}_{\mathbf{w}})$ to be the (infinite) directed graph whose vertices are the partial triangulations of the decorated surface $\overline{\mathbf{S}}_{\mathbf{w}}$ and whose edges are given by forward flips.

Definition 3.2. *Let $\mathcal{V} \subset \mathcal{D}$ be a thick triangulated subcategory. We say that a heart $\mathcal{A}$ of $\mathcal{D}$ is* $\mathcal{V}$-compatible*, if $\mathcal{A} \cap \mathcal{V}$ is a Serre subcategory of $\mathcal{A}$.*

We call a heart $\overline{\mathcal{A}}$ of $\mathcal{D}/\mathcal{V}$ of quotient type *if there is a $\mathcal{V}$-compatible heart $\mathcal{A}$ of $\mathcal{D}$ whose essential image in $\mathcal{D}/\mathcal{V}$ is $\overline{\mathcal{A}}$.*

We define the *principal component* $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_{\mathbf{w}})$ to be the full subgraph of partial triangulations that admit a refinement to a triangulation in $\mathrm{EG}^\circ(\mathbf{S}_\Delta)$, i.e. the full subgraph given by triangulations reachable by a finite number of flips from $\mathbb{T}_0$. We define the *principal component* $\mathrm{EG}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ to be the full subgraph of $\mathrm{EG}(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ consisting of hearts of quotient type that admit a representative in the distiguished component $\mathrm{EG}^\circ(\mathcal{D}(\mathbf{S}_\Delta))$. It is a priori not clear that these definitions yield connected components. This is proven along with [BMQS22, Theorem 5.9], which moreover states that

$$\mathrm{EG}^\bullet(\overline{\mathbf{S}}_{\mathbf{w}}) \;\cong\; \mathrm{EG}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}})) \tag{7}$$

and that both graphs are (m, m)-regular.

We now define the *principal component of the stability manifold* $\mathrm{Stab}(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ to be

$$\mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}})) \;=\; \mathbb{C} \cdot \bigcup_{\mathcal{H} \in \mathrm{EG}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))} \mathrm{Stab}(\mathcal{H}). \tag{8}$$

The terminology is justified by the following results:

Proposition 3.3 ([BMQS22])**.** *The space* $\mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ *is union of connected components of* $\mathrm{Stab}(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$*.*

Referring to Section 6 for the definition of framed quadratic differentials we recall here the generalization of the Bridgeland-Smith isomorphism that serves as motivation for definition of multi-scale stability conditions using the comparison to compactification of strata, see Section 6.

Theorem 3.4 (Theorem 1.1 of [BMQS22])**.** *There is an isomorphism of complex manifolds*

$$K : \mathrm{FQuad}^\bullet(\overline{\mathbf{S}}_{\mathbf{w}}) \to \mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$$

between the principal part of the space of Teichmüller-framed quadratic differentials and the principal part of the space of stability conditions on $\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}})$.

3.3. **Braid groups and spherical twists.** Motivated by the relation to groups of autoequivalences given in (17) and (18) below we recall a few basic properties of the braid groups B_{n+1} on $n+1$ strands, see e.g. [FM12, Section 9]. The standard generators are the τ_i twisting the strands i and $i+1$. They satisfy the defining *standard braid relations*.

$$\tau_i\tau_j\tau_i \;=\; \tau_j\tau_i\tau_j \quad \text{if} \quad |i-j| = 1, \quad \text{and} \quad \tau_i\tau_j \;=\; \tau_j\tau_i \qquad \text{if} \quad |i-j| \geq 2, \tag{9}$$

The *pure braid group* is the kernel of the homomorphism recording the strand permutation, i.e. sits in the exact sequence

$$0 \to \mathrm{PB}_{n+1} \to B_{n+1} \to S_{n+1} \to 0\,. \tag{10}$$

The center of B_{n+1} is cyclic, generated by the element

$$\theta_n = (\tau_1\tau_2\cdots\tau_n)^{n+1} \tag{11}$$

except for the case where of $n = 1$, since B_2 is cyclic where thus θ_2 is the square of the generator. In all cases the element θ_n as defined above also belongs to the pure braid group PB_{n+1}. As a special case of the Birman exact sequence we obtain

$$1 \to F_{n+1} \to \mathrm{PB}_{n+1} \to \mathrm{PB}_n \to 1\,, \tag{12}$$

where F_{n+1} is the free group on $n+1$ generators. This exact sequence is split by adding the extra strand. Iterating this we obtain for each r consecutive integers a natural homomorphism

$$\varphi_{r,n} : \mathrm{PB}_{r+1} \to \mathrm{PB}_{n+1} \tag{13}$$

Via this homomorphism we define for $I = \{1,\dots,r\}$ the elements $\theta_{I,n} := \varphi_{r,n}(\theta_r) \in B_{n+1}$. These correspond to a full rotation of a disc encircling precisely the points in I.

More generally we will define the braid group B_Q associated with a quiver Q, see [Qiu16, Definition 10.1 and Proposition 10.4]. It is generated by an element $\widetilde{\tau}_i$ for each vertex of the quiver and defined by the relations

$$\begin{aligned} \widetilde{\tau}_i\widetilde{\tau}_j\widetilde{\tau}_i \;&=\; \widetilde{\tau}_j\widetilde{\tau}_i\widetilde{\tau}_j \quad \text{if} \quad |i-j| = 1 \\ \widetilde{\tau}_i\widetilde{\tau}_j \;&=\; \widetilde{\tau}_j\widetilde{\tau}_i \quad \text{if} \quad |i-j| \ge 2, \\ R_i = R_j &\quad \text{for each cycle } 1 \to 2 \to \cdots \to m \to 1\,, \end{aligned} \tag{14}$$

where $R_i = \widetilde{\tau}_i\widetilde{\tau}_{i+1}\cdots\widetilde{\tau}_m\widetilde{\tau}_1\cdots\widetilde{\tau}_{i-1}$. In the case $Q = A_n$ we retrieve the above definition of $B_{A_n} = B_{n+1}$. We will be most interested in the case that Q is of type A_n though not necessarily equal to A_n. Suppose the vertices in the index set I form a subquiver of type A_r. Then the braid group of the restricted quiver $B_{Q_I} \cong B_{r+1}$ and we let θ_r be its central element. Inclusion of strands again defines a natural homomorphism $\varphi_{I,n} : B_{Q_I} \to B_{n+1}$ and we define in this more general context $\theta_{I,n} := \varphi_I(\theta_r)$.

Given a quiver (Q, W) and its Ginzburg algebra Γ, we let $\mathrm{ST}(\Gamma) \le \mathrm{Aut}(\mathrm{pvd}(\Gamma))$ be the *spherical twist group* (see Seidel-Thomas [ST01]) of $\mathrm{pvd}(\Gamma)$, that is the subgroup generated by the set of spherical twists Φ_S for all simples S of Γ, where the *twist functor* Φ_S is defined by

$$\Phi_S(X) \;=\; \mathrm{Cone}\,(S \otimes \mathrm{Hom}^\bullet(S, X) \to X)\,. \tag{15}$$

This uses that in the case $\mathcal{D} = \mathcal{D}^3_Q$, as a consequence of [KQ15, Corollary 8.5], all the simples in any heart of $\mathcal{D}$ are spherical.

Remark 3.5. For a heart $\mathcal{A}$ with simples $S_1, ..., S_n$ listed in an order such that $\dim(\mathrm{Ext}^1(S_j, S_i)) = 0$ for $j < i$ and $\dim(\mathrm{Ext}^1(S_i, S_j)) = 0$ for $j > i$ the mutated heart $\mu^\sharp_{S_i}(\mathcal{A})$ has simples

$$S_1, ..., S_{i-1}, S_i[1], \Phi^{-1}_{S_i}(S_{i+1}), ..., \Phi^{-1}_{S_i}(S_n)\,,$$

compare [BS15, Proof of Proposition 7.1]. Moreover,

$$\mu^\sharp_{S[1]}\mu^\sharp_S\mathcal{A} \;=\; \Phi^{-1}_S\mathcal{A}\,. \tag{16}$$

We write $\mathrm{ST}(A_n)$ for spherical twist group of a quiver of type A_n.

Proposition 3.6 ([ST01; Qiu16; Qiu18]). *There is an isomorphism* $\mathrm{ST}(\Gamma) \cong B_Q$ *between the twist groups, sending the standard generators* $\tau_i \to \Phi_{S_i}$ *to the standard generators. In particular the group* $\mathrm{ST}(A_n)$ *is isomorphic to the braid group* B_{n+1}.

We now apply this to understand the groups of autoequivalences for $\mathcal{D} = \mathrm{pvd}(\Gamma)$. By [BS15, Theorem 9.9] there is an exact sequence

$$1 \to \mathcal{ST}(\mathcal{D}^3_Q) \to \mathcal{A}ut^\circ(\mathcal{D}^3_Q) \to \mathrm{MCG}(\mathbf{S}) \to 1, \tag{17}$$

where $\mathcal{ST}(\mathcal{D}^3_Q)$ is the quotient of $\mathrm{ST}(\mathcal{D}^3_Q)$ by its subgroup of negligible automorphisms. In the special case $\mathcal{D}^3_Q = D^3_{A_n}$ the mapping class group is $\mathrm{MCG}(\overline{\Delta}, \mathbb{M}_{n+3}) \cong \mathbb{Z}/(n+3)$, so there is an exact sequence

$$1 \to B_{n+1} \to \mathcal{A}ut^\circ(\mathcal{D}^3_{A_n}) \to \mathbb{Z}/(n+3) \to 1 \tag{18}$$

see e.g. [BS15, Section 12.1],

In Section 5.3 we need the following action on Grothedieck groups to control the effect of the action of $\theta_{I,n}$.

Lemma 3.7 ([Ike17, Section 4]). *On* $K(\mathcal{D}^3_Q)$, *a spherical twist* Φ_{S_i} *induces a group homomorphism* $[\Phi_i]$ *defined by*

$$[\Phi_i]([E]) \;=\; [E] - \chi([S_i],[E])[S_i], \tag{19}$$

where $\chi(\cdot,\cdot)$ *denotes the Euler pairing.*

4. Multi-scale stability conditions

We start with a definition that makes sense for general triangulated categories with finite rank Grothendieck group. We will then define a $\mathbb{C}$-action and a notion of plumbing stability conditions that will be used to give a topology on the set of multi-scale stability conditions. In all these steps we have to be much more restrictive, essentially restricting to $\mathcal{D} = \mathcal{D}^3_{A_n}$. We indicate the technical difficulties needed to overcome in order to generalize to CY_3-quiver categories or beyond.

Definition 4.1. *Let* $\mathcal{D}$ *a triangulated category with* $\mathrm{rank}(K(\mathcal{D})) < \infty$. *A* multi-scale stability condition *on* $\mathcal{D}$ *consists of an equivalence class of the following data:*

- *a* multi-scale heart $\mathcal{A}_\bullet = (\mathcal{A}_i)$, *i.e., a collection* $\mathcal{A}_L \subset \cdots \mathcal{A}_1 \subset \mathcal{A}_0$ *of abelian categories, and*
- *a* multi-scale central charge, *i.e., a collection* $Z_\bullet = (Z_i)_{i=0}^L$ *of non-zero* $\mathbb{Z}$*-linear maps on the Grothendieck groups* $Z_i : K(\mathcal{A}_i) \to \mathbb{C}$,

with the property that

- *the abelian category* $\mathcal{A}_i$ *is generated by the non-zero objects* E *in* $\mathcal{A}_{i-1}$ *with* $Z_{i-1}(E) = 0$ *for all* $i \geq 1$,
- *the abelian category* $\mathcal{A}_i$ *is a heart of* $\mathcal{V}_i$, *which is defined as* $\mathcal{V}_0 = \mathcal{D}$ *and for all* $i \geq 1$ *as the thick triangulated subcategory of* $\mathcal{V}_{i-1}$ *generated by* $\mathcal{A}_i$,
- *the map* Z_i *factors through* $K_{i-1} := \mathrm{Ker}(Z_{i-1})$,
- *the induced heart* $\overline{\mathcal{A}_i} = \mathcal{A}_i/\mathcal{A}_{i+1}$ *together with the induced central charge* $\overline{Z_i} : K(\mathcal{V}_i/\mathcal{V}_{i+1}) \to \mathbb{C}$ *form a stability condition in the usual sense on* $\mathcal{V}_i/\mathcal{V}_{i+1}$ *for all* $i = 0, \ldots, L$.

Two multi-scale stability conditions $(\mathcal{A}_\bullet, Z_\bullet)$ *and* $(\mathcal{A}'_\bullet, Z'_\bullet)$ *are* equivalent, *if*

i) there is equality of triangulated categories $\mathcal{V}_i = \mathcal{V}'_i$ *for* $i = 0, \ldots L$, *ii) the induced stability conditions* $(\overline{A_i}, \overline{Z}_i)$ *and* $(\overline{A_i}', \overline{Z}_i')$ *are projectively equivalent for* $i = 1, \ldots, L$*, and are equal for* $i = 0$.

Two multi-scale stability conditions $(\mathcal{A}_\bullet, Z_\bullet)$ *and* $(\mathcal{A}'_\bullet, Z'_\bullet)$ *are* projectively equivalent *if the projective equivalence in ii) above holds for* $i = 0, \ldots, L$.

We write $[\mathcal{A}_\bullet, Z_\bullet]$ for an equivalence class, and $(\mathcal{A}_\bullet, Z_\bullet)$ for a representative of a multi-scale stability condition. Moreover we denote by $\mathcal{V}_\bullet$ the collection of nested triangulated subcategories $(\mathcal{V}_i)$ defined by $(\mathcal{A}_\bullet, Z_\bullet)$. Sometimes write $\mathcal{V}_i^Z$ for the categories $\mathcal{V}_i$ defined above to indicate the dependence on $Z_\bullet$.

The definition relies on the following lemma for the quotient hearts to be meaningful.

Lemma 4.2. *The subcategory* $\mathcal{A}_{i+1}$ *is Serre in* $\mathcal{A}_i$ *and* $\mathcal{A}_{i+1} = \mathcal{V}_{i+1} \cap \mathcal{A}_i$ *for all i, i.e.,* $\mathcal{A}_{i+1}$ *is* $\mathcal{V}_{i+1}$*-compatible in the sense of Definition 3.2. In particular, the inclusion* $\iota : \mathcal{V}_{i+1} \to \mathcal{V}_i$ *is t-exact with respect to* $\mathcal{A}_{i+1}$ *and* $\mathcal{A}_i$*, and* $\mathcal{A}_i$ *induces a quotient heart in* $\mathcal{V}_i/\mathcal{V}_{i+1}$*. Moreover,* $K(\mathcal{V}_i/\mathcal{V}_{i+1}) = K_i/K_{i+1}$ *so that* Z_i *descends to* $\overline{Z_i}$*, as required.*

Proof. Serreness of $\mathcal{A}_{i+1} \subset \mathcal{A}_i$ follows from the additivity of Z_i on short exact sequences and the fact that Z_i takes values in a strictly convex sector in $\mathbb{C}$. The second statement follows from the observation that $Z_i(X) = 0$ for any $X \in \mathcal{V}_{i+1}$. Serreness of $\mathcal{A}_{i+1} \subset \mathcal{A}_i$ guarantees that $\mathcal{V}_{i+1}$ consists on objects of $\mathcal{V}_i$ whose cohomology with respect to $\mathcal{A}_i$ is concentrated in $\mathcal{A}_{i+1}$, and that the t-structure restricts, so the next claim follows from [AGH19, Proposition 2.20] or [CR17, Lemma 3.3]. For the last observe that K_{i+1} is the image of $\iota_* : K(\mathcal{A}_{i+1}) \to K_i$ and that the right exact sequence $K(\mathcal{A}_{i+1}) \to K(\mathcal{A}_i) \to K(\mathcal{A}_i/\mathcal{A}_{i+1}) \to 0$ is used to compute the Grothendieck group of the quotient category. □

Let $\mathrm{MStab}(\mathcal{D})$ be the set of all multi-scale stability conditions on $\mathcal{D}$. The integer L in Definition 4.1 will be referred to as the number of *levels below zero* of the stability condition. A usual stability condition has $L = 0$.

Reachability. We now fix a component $\mathrm{Stab}^\circ(\mathcal{D})$ of the stability manifold of $\mathcal{D}$. If $\mathcal{D} = \mathcal{D}^3_{Q_3}$ is a quiver category we use an initial triangulation $\mathbb{T}_0$ of a DMS as in Section 3.1 to single out this components.

A multi-scale stability condition $(\mathcal{A}_\bullet, Z_\bullet)$ is called *reachable* if the top level heart $\mathcal{A}_0$ supports stability conditions in $\mathrm{Stab}^\circ(\mathcal{D})$. We denote the set of all reachable multi-scale stability conditions on $\mathcal{D}$ by $\mathrm{MStab}^\circ(\mathcal{D})$, and the set of reachable multi-scale stability conditions with the same $\mathcal{V}_\bullet$ by $\mathrm{MStab}^\circ(\mathcal{D}, \mathcal{V}_\bullet)$.

In the next section, we will informally call a multi-scaled stability conditions with at least one level below zero a *boundary point*, and finally prove this is actually the case.

Groups of autoequivalences. For general $\mathcal{D}$ we define the group $\mathrm{Aut}(\mathcal{D}, \mathcal{V})$ to be the autoequivalences that stabilize $\mathcal{V}$. For $\mathcal{D} = \mathcal{D}^3_Q$ we use bullets to denote those autoequivalences that moreover stabilize the principal components defined in Section 3.2.

Lemma 4.3. *The factor group*

$$\mathcal{A}ut_{\mathrm{lift}}(\mathcal{D}/\mathcal{V}) := \mathcal{A}ut^\bullet(\mathcal{D}, \mathcal{V})/\mathcal{A}ut^\bullet(\mathcal{V}) \tag{20}$$

acts properly discontinuously on $\mathrm{Stab}^\bullet(\mathcal{D}/\mathcal{V})$.

The stabilizer $H_{[\sigma]} \subset \mathcal{A}ut_{\mathrm{lift}}(\mathcal{D}/\mathcal{V})$ of a projectivized stability condition $[\sigma]$ is a finite extension of a subgroup Z in the center of $\mathcal{A}ut_{\mathrm{lift}}(\mathcal{D}/\mathcal{V})$. In case $\mathcal{D} = \mathcal{D}^3_{A_n}$ the group Z is the center, generated by θ_n defined in (11).

Proof. The first statement is shown in the proof of [BMQS22, Theorems 8.1, 8.2], summarized here in Theorem 3.4, (see in particular the part about the orbifold structure, together with Equation (5.7)).

For the second statement we may pass, thanks to the first statement, to the finite index subgroup of the stabilizer $H_{[\sigma]}$ of a projective stability condition that acts trivially on a neighborhood of the unprojectivized σ. As in the proof of [BMQS22, Theorems 8.2], since the action of $\mathbb{C}$ and $\mathcal{A}ut_{\mathrm{lift}}(\mathcal{D}/\mathcal{V})$ commute, the second statement follows. The last claim is a restatement of braid group properties from Section 3.3. ☐

4.1. **Numerical data of multi-scale stability conditions of type A_n.** From here on we restrict to the case $\mathcal{D} = \mathcal{D}^3_{A_n}$. In this case we can completely describe the subcategories and hearts that appear in a multi-scale stability condition.

Lemma 4.4. *If $(\mathcal{A}_\bullet, Z_\bullet)$ is a multi-scale stability condition and $\mathcal{A}_0 = \operatorname{mod} \mathcal{J}_Q$, then $\mathcal{A}_1 = \operatorname{mod} \mathcal{J}_{Q_I}$ where I is a subset of the vertices of $\mathcal{Q}$, and $\mathcal{V}_1 = \operatorname{pvd} \Gamma_I$.*

Moreover, there is a bijection of the subcategories $\mathcal{V}_1^Z$ with homotopy classes of decorated marked subsurfaces $\Sigma \subset \mathbf{S}_\Delta$, such that each component Σ_j for $j \in J$ is of type A_{n_j}, i.e. a disc with n_j+1 decoration points in its interior and n_j+3 marked points at its boundary. Here $n_j \geq 1$ and the decomposition is constrained precisely by $\sum_{j\in J}(n_j+1) \leq n+1$, where equality is allowed if and only if $|J| \geq 2$.

Using this notation we say that $\mathcal{V}_1^Z$ is a subcategory *of type* $\rho := (n_1, \dots, n_{|J|})$. Iterating this over all $\mathcal{V}_i$ appearing in a multi-scale stability condition, we say that $[\mathcal{A}_\bullet, Z_\bullet]$ *is of type* $\boldsymbol{\rho} = (\rho_i)_{i=1}^L$, where ρ_i is the type of $\mathcal{V}_i$.

Proof. Consider $\mathcal{A}_1 \subset \mathcal{V}_1^Z$. Since $\mathcal{A}_1 \subset \mathcal{A}_0$ is Serre, it is generated by a subset $\mathcal{S}_1 \subset \operatorname{Sim}(\mathcal{A}_0)$ of the simples of $\mathcal{A}_0$, those whose Z_0-image is zero. By the correspondence summarized in Section 2.4 this defines the subquiver Q_I and shows $\mathcal{A}_1 = \operatorname{mod} \mathcal{J}_I$.

For the second claim, we show the chain of equalities

$$\mathcal{V}_1 \;=\; \operatorname{pvd}_{J_I}(\Gamma) \;=\; \operatorname{pvd}_{\Gamma/\Gamma e\Gamma}(\Gamma) \;=\; \operatorname{pvd}(\Gamma_I)\,,$$

where $\operatorname{pvd}_R(\mathcal{D}) \subset \operatorname{pvd}(\mathcal{D})$ is the full subcategory with cohomologies in $\operatorname{mod} R$. The first follows using the characterization of $\mathcal{A}_0$ as a heart in terms of a decomposition of objects in $\operatorname{pvd}_{J_I}(\Gamma)$ into triangles and Serreness of $\operatorname{mod} \mathcal{J}_I$ in $\mathcal{A}_0$, as in Lemma 4.2. For the second we just intersect $\Gamma_I = \Gamma/\Gamma e\Gamma$ with $H^0\Gamma = \mathcal{J}$. The last equality follows from [KY18, Corollary 6.4 (b)], where we can take $B = \Gamma/\Gamma e\Gamma = \Gamma_I$ thanks to Theorem 7.1 in loc. cit.

Conversely, choosing Z_0 to be zero for any subset of the simples and $\overline{\mathbb{H}}$-valued for the complementary set of simples defines a subcategory $\mathcal{V}_1^Z$ that can be be completed to a multi-scale stability condition.

We now translate into the language of Section 3. Let $\mathbb{T}$ be the triangulation of the DMS $\mathbf{S}_\Delta$ corresponding to $\mathcal{A}_0$. Dual to the open arcs forming the triangulation there are closed arcs connecting the decorating points. These are in canonical bijection to the simples in $\mathcal{A}_0$, see e.g. the summary in [BMQS22, Theorem 7.2]. Let $\mathbb{A}_1^\vee = \{\eta_S, S \in \mathcal{S}_1\}$ be the closed arcs corresponding to $\mathcal{A}_1$ and $\mathbb{A}_1$ the set of dual closed arcs. Let $\Sigma = \Sigma_1$ be the subsurface consisting of a tubular neighborhood of $\mathbb{A}_1^\vee$ and let $\Sigma^{(j)}$ denote its connected components. They are all homotopic to a disc containing a certain number, say n_j+1, of simple zeros with one boundary component. Homotoping the open arcs in $\mathbb{A}_0$ so that they intersect Σ minimally, we deduce from duality that precisely those in $\mathbb{A}_1$ have non-trivial intersection with Σ. We may thus mark $\kappa^{(j)} = n_j+3$ points on the boundary of $\Sigma^{(j)}$ and restrict the arcs in $\mathbb{A}_1$ to arcs in Σ connecting these boundary points so that $\mathbb{A}_1|_\Sigma$ is a triangulation of Σ. In fact the quiver associated with each subsurface $\Sigma^{(j)}$ is of type A_{n_j}.

The constraints for n_j reflect that the total number of decoration points in $\mathbf{S}_\Delta$ has to be at least two and the fact that there is at least one simple outside $\mathcal{A}_1$, i.e. a closed arc not contained in Σ. This arc has to connect Σ with a decoration point outside Σ (the case of strict inequality) or connects two components (the case $|J|\geq 2$). $\square$

As a consequence we may associate with each subsurface $\Sigma_j^{(i)}$ a subcategory $\mathcal{V}_i^{(j)}$ of $\mathcal{V}_i$, that jointly give an orthogonal decomposition of $\mathcal{V}_i$. We refer to $\mathcal{V}_i^{(j)}$ as the *components* of $\mathcal{V}_i$.

Recall the notion of principal components from Section 3.2.

Corollary 4.5. *Suppose that $(\mathcal{A}_\bullet, Z_\bullet) \in \mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})$. Then the quotient hearts $\overline{\mathcal{A}}_i \subset \mathcal{V}_i/\mathcal{V}_{i+1}$ support a stability condition in the principal component $\mathrm{Stab}^\bullet(\mathcal{V}_i/\mathcal{V}_{i+1})$ of the stability manifold of $\mathcal{V}_i/\mathcal{V}_{i+1}$ for all $i=0,\dots,L$.*

Proof. In fact they belong to the one corresponding to the triangulation $\mathbb{A}_i|_{\Sigma_i}$, extending the notation of the previous proof in the obvious way. $\square$

4.2. **The $\mathbb{C}$-action.** Extending the $\mathbb{C}$-action from usual stability conditions to multi-scale stability conditions is a crucial ingredient for the subsequent plumbing construction.

Proposition 4.6. *Recall that we suppose $\mathcal{D}=\mathcal{D}^3_{A_n}$. Then there is an action of $\mathbb{C}$ denoted by $(\lambda,[\mathcal{A}_\bullet, Z_\bullet]) \mapsto \lambda\cdot[\mathcal{A}_\bullet, Z_\bullet] =: [\mathcal{A}'_\bullet, Z'_\bullet]$ on $\mathrm{MStab}^\circ(\mathcal{D})$, such that*

(i) the collection of subcategories $\mathcal{V}_i^Z = \mathcal{V}_i^{Z'} =: \mathcal{V}_i$ is preserved,
(ii) $Z'_i = e^{-\sqrt{-1}\pi\lambda} Z_i$, and
(iii) $\lambda.(\overline{\mathcal{A}}_i, \overline{Z}_i) = (\overline{\mathcal{A}}'_i, \overline{Z}'_i)$ is the usual $\mathbb{C}$-action on $\mathrm{Stab}(\mathcal{V}_i/\mathcal{V}_{i+1})$.

Thanks to Proposition 4.6, proven below, we define the projectivized space of multi-scale stability conditions $\mathbb{P}\mathrm{MStab}^\circ(\mathcal{D}) = \mathrm{MStab}^\circ(\mathcal{D})/\mathbb{C}$. In this language note that retaining just the filtration steps from a level j onward, i.e. the datum of tuples $(\mathcal{A}_{\geq j}, Z_{\geq j})$ together with the ambient triangulated category $\mathcal{V}^Z_{j-1}$ gives by definition an element in $\mathbb{P}\mathrm{MStab}^\circ(\mathcal{V}^Z_{j-1})$ with $L-j+1$ levels below zero.

The restriction $\mathcal{D}=\mathcal{D}^3_{A_n}$ stems from two requirements in the proof. First, we need finite type. Second, we need a way to lift tilts from quotient categories to $\mathcal{D}$ itself. We have shown this for quiver categories in [BMQS22] and isolate this step in the following notion.

Definition 4.7. *Given a thick triangulated subcategory $\mathcal{V}\subset\mathcal{D}$, a heart $\mathcal{A}$ of $\mathcal{D}$ and a simple object S in $\mathcal{A}\setminus(\mathcal{V}\cap\mathcal{A})$, we call a heart $\mathcal{A}'$ with $\overline{\mathcal{A}} = \overline{\mathcal{A}'} := \mathcal{A}'/(\mathcal{V}\cap\mathcal{A}')$ a* convenient representative *with respect to (the forward tilt at) S if $\mathcal{A}'$ is $\mathcal{V}$-compatible and if*

$$\mathrm{ext}^1(T,S) := \dim(\mathrm{Ext}^1(T,S)) = 0 \quad \textit{for all simples} \quad T\in\mathcal{V}\cap\mathcal{A}'. \tag{21}$$

It means that a simple tilt of $\mathcal{A}$ at S induces a simple tilt of $\overline{\mathcal{A}}$ at $\overline{S}$.

Lemma 4.8. *Suppose $\mathcal{D} = \mathcal{D}^3_Q$. For every $\mathcal{V}$-compatible finite heart $\mathcal{A}$ and every simple S there exists a convenient representative $\mathcal{A}'$, which can be obtained from $\mathcal{A}$ by a finite sequence of simple tilts at simples in $\mathcal{A} \cap \mathcal{V}$.*

If $\mathcal{A}'$ is a convenient representative for S, then $\mu^\sharp_S \mathcal{A}'$ is $\mathcal{V}$-compatible and

$$\overline{\mu^\sharp_S \mathcal{A}'} \;=\; \mu^\sharp_S \overline{\mathcal{A}'}\,. \tag{22}$$

Proof. See [BMQS22, Proposition 5.8]. □

In the following we will give an explicit procedure for finding a convenient representative if $\mathcal{D} = \mathcal{D}^3_{A_n}$, that will be useful later.

Lemma 4.9. *Recall that we suppose $\mathcal{D} = \mathcal{D}^3_{A_n}$. Let $\mathcal{A}$ be a heart compatible with $\mathcal{V}$ and let $S_0 \in \mathcal{A} \backslash \mathcal{A} \cap \mathcal{V}$ be simple. Then there exist a (possibly empty) set of indecomposables $\{S_1, \dots, S_{1\dots m}, S'_1, \dots, S'_{1\dots m'}\} \subset \mathcal{A} \cap \mathcal{V}$ explicitly defined in the proof below, such that $(\mu^\sharp_{S'_{1\dots m'}} \cdots \mu^\sharp_{S'_1})(\mu^\sharp_{S_{1\dots m}} \cdots \mu^\sharp_{S_1})(\mathcal{A})$ is a convenient representative of $\overline{\mathcal{A}}$ with respect to S_0.*

Proof. Observe that for any simple $S \in \mathcal{A}$ the number of (isomorphism classes of) simples T in $\mathcal{A}$ satisfying $\operatorname{ext}^1(T, S) = 1$ is at most 2, since $\mathcal{A} = \operatorname{rep}(Q, W)$ with (Q, W) a quiver of A_n-type or, more generally, since it comes from a triangulation of a surface. If there are no simples in $\mathcal{A} \cap \mathcal{V}$ with this property, then $\mathcal{A}$ is convenient with respect to S_0 and we are done. Otherwise, we fix a simple $S_1 \in \mathcal{A} \cap \mathcal{V}$ with

$$\operatorname{ext}^1(S_1, S_0) \;=\; 1, \tag{23}$$

and we define $S_1, \dots, S_m$ as the maximal collection of simples in $\mathcal{A} \cap \mathcal{V}$, with

$$\operatorname{ext}^1(S_i, S_{i-1}) \;=\; 1 \quad \text{and} \quad \operatorname{ext}^1(S_{i+1}, S_{i-1}) \;=\; 0 \tag{24}$$

for $i \geq 1$. Note that the second condition in (24) singles out exactly one among the two possible objects with non-trivial extension with S_i. Consequently the collection is uniquely specified for a given $\mathcal{A}$, and the definition is well-posed. See Figure 2 for an example of the corresponding ext-quiver.

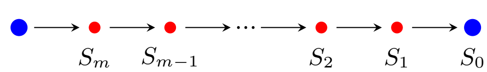


FIGURE 2. (Partial) ext-quiver containing the A_{m+1}-configuration of $S_0, S_1, \dots, S_m$ defined in the text. The small red dots correspond to simples in $\mathcal{A} \cap \mathcal{V}$, while the big blue dots correspond to simples of $\mathcal{A}$ not in $\mathcal{V}$.

Tilting $\mathcal{A}$ at S_1 produces a configuration of simples in $\mu^\sharp_{S_1}\mathcal{A} \ni S_0$ such that the sequence of objects defined by (23),(24) in $\mu^\sharp_{S_1}\mathcal{A}$ has length $m-1$ and consists on $S_{12}, S_3, \dots, S_m$, as displayed in Figure 3 using the correspondence between simple tilts and mutations.

Tilting inductively at $S_{1\dots i}$ (recall the notation from (3)), the procedure leads to a configuration where such a sequence has length 0, as desired, see Figure 4. We

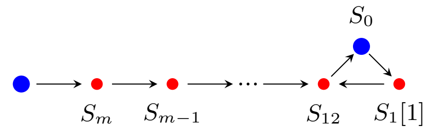


FIGURE 3. Mutation at S_1 of the ext-quiver of Figure 2.

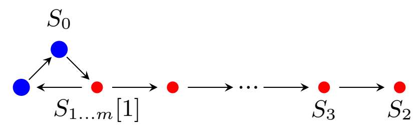


FIGURE 4. The result of mutating at $S_1, S_{12}, \ldots, S_{1\ldots m}$ the ext-quiver of Figure 2.

define $\mathcal{X} = \langle S_{1\ldots m}, \ldots, S_1 \rangle$ so that $\mu^\sharp_{\mathcal{X}} := \mu^\sharp_{S_{1\ldots m}} \cdots \mu^\sharp_{S_1}$ by Proposition 2.2. By construction, $\mathcal{X} \subset \mathcal{A} \cap \mathcal{V}$ and $\overline{\mu^\sharp_{\mathcal{X}}\mathcal{A}} = \overline{\mathcal{A}} \subset \mathcal{D}/\mathcal{V}$.

Suppose now that there exists another simple $S'_1 \neq S_1$ satisfying (23). Proceeding in the same way we define S'_i using (23) and (24) and define inductively $S'_{1\ldots i}$ for $i = 1, \ldots, m'$ as above. They are not in $\langle S_1, \ldots, S_{m+1} \rangle$, due to the second condition in (24). Then, with $\mathcal{X}' = \langle S'_{1\ldots m}, \ldots, S'_1 \rangle$,

$$\mu^\sharp_{\mathcal{X}'} \mu^\sharp_{\mathcal{X}} (\mathcal{A}) = (\mu^\sharp_{S'_{1\ldots m'}} \cdots \mu^\sharp_{S'_{12}} \mu^\sharp_{S'_1})(\mu^\sharp_{S_{1\ldots m}} \cdots \mu^\sharp_{S_{12}} \mu^\sharp_{S_1})(\mathcal{A})$$

is a convenient representative of $\overline{\mathcal{A}}$ with respect to S_0. $\square$

Proof of Proposition 4.6. Here and in many cases in the sequel, all aspects of the proof are visible in the situation with just two levels, i.e., $L = 1$, and for expository simplicity we restrict to this case and if needed we mention briefly in the end how to proceed by induction. Suppose $\operatorname{Re} \lambda \geq 0$, the other case is analogous.

For a rescaling by $e^{-\pi i \lambda}$ with $\lambda \in i\mathbb{R}$ and for a rotation ($\lambda \in \mathbb{R}$) so that the phase of no simple in $\mathcal{A}_0$ with non-zero central charge exceeds $(0, 1]$, we just apply (ii) to the multi-scale central charge and all the other conditions still hold. It thus suffices to consider general rotations, i.e. $\lambda \in (0, 1]$, repeating the process $\lfloor \lambda \rfloor$ many times.

We denote by $\mathcal{F}^1_\lambda \subset \mathcal{A}_1$ the torsion-free class induced by the action of λ on $(\mathcal{A}_1, Z_1)$. Similarly we let $\overline{\mathcal{F}}_\lambda \subset \overline{\mathcal{A}_0}$ be the analogous torsion-free class for the λ-action on $(\overline{\mathcal{A}_0}, \overline{Z_0})$. We can decompose the tilt at $\overline{\mathcal{F}}_\lambda$ as a composition of tilts at finitely many simple torsion-free classes $\overline{\mathcal{F}}_i = \langle \overline{X_i} \rangle \subset \overline{\mathcal{A}_0}$ according to Proposition 2.2.

The subcategory $\mathcal{F}^1_\lambda$ is a torsion-free class in $\mathcal{A}_0$ and we first forward-tilt $\mathcal{A}_0$ and $\mathcal{A}_1$ at $\mathcal{F}^1_\lambda$. Then we inductively "lift" the simple tilt at $\overline{X_i}$ at $\mu^\sharp_{\overline{\mathcal{F}}_{i-1}} \overline{\mathcal{A}_0}^{(i-1)}$ (with $\overline{\mathcal{A}_0}^{(0)} = \overline{\mathcal{A}_0}$) to the upper level in the following way. If $S = X_1$, we apply Lemma 4.8 and forward tilt $\mathcal{A}_0$ to arrive at a convenient representative $\mathcal{A}'_0 = \mu^\sharp_{\mathcal{X}} \mathcal{A}_0$ for S. Second, we tilt forward at S, and third, we perform the backward tilt at $\mathcal{X}[1]$. At the end we arrive at a heart $\mathcal{A}''_0$ on which Z_0 is still well-defined, and with the following properties:

(1) $\mathcal{V}''_0 = \mathcal{V}_0$, since after each of the three steps the simples annihilated by Z_0 generate the same category;

(2) the quotient heart $\overline{\mathcal{A}}_0''$ coincides with $\mu^\sharp_S\overline{\mathcal{A}_0}$, thanks to (22);
(3) the intersection $\mathcal{A}_0''\cap\mathcal{V}_0=\mathcal{A}_1$, since the forward and backward tilts cancel on $\mathcal{A}_1$.

Repeating this process for all i, in the end we change the central charge as required by (ii). Using (1) at each step ensures (i), and (2) together with (3) at each step ensure (iii). This procedure indeed defines an action of $\mathbb{C}$, since the equivalence class of the multi-scale stability condition is uniquely determined by the conditions (i)–(iii). It obviously agrees with the $\mathbb{C}$-action on $\mathrm{Stab}(\mathcal{V}_0/\mathcal{V}_1)$. $\square$

For later use we record that the "lifts" of the tilts at $\overline{\mathcal{F}}_i$ used in the previous proof are actually tilts at explicit torsion-free classes $\mathcal{F}_i$.

Lemma 4.10. *Recall that $\mathcal{D}=\mathcal{D}_{A_n}$, let $\mathcal{A}_\bullet=(\mathcal{A}_0,\mathcal{A}_1)$ and $Z_\bullet=(Z_0,Z_1)$. For $\lambda\in\mathbb{R}_{\geq 0}$ such that*

- *$\lambda\cdot(\overline{\mathcal{A}_0},\overline{Z_0})=(\mu^\sharp_{S_0}\overline{\mathcal{A}_0},e^{-\pi\lambda}\overline{Z_0})$ in $\mathrm{Stab}^\circ(\mathcal{D}/\mathcal{V})$, and*
- *there are no indecomposables in $\mathcal{A}_1$ with phase $\phi_{\overline{Z_1}}$ less than or equal to λ,*

the action by λ on $[\mathcal{A}_\bullet,Z_\bullet]$ gives $[\mathcal{A}'_\bullet,Z'_\bullet]$ with nested hearts

$$\mathcal{A}_0' \;=\; \mu^\sharp_{\mathcal{F}}\mathcal{A}_0,\quad \mathcal{A}_1' \;=\; \mathcal{A}_1,$$

for $\mathcal{F}=\langle S'_{01\ldots m'},\ldots S'_{01},S_{01\ldots m},\ldots S_{01},S_0\rangle$, using the same notation as in proof of Lemma 4.9. Moreover $\mathcal{F}\subset\langle S_0,\mathcal{A}_1\rangle\setminus\mathcal{A}_1$.

Proof. Suppose for simplicity $m>0$, $m'=0$ in the notation of Lemma 4.9. We know that $\mathcal{A}_0'=\mu^\flat_{\mathcal{X}[1]}\mu^\sharp_{S_0}\mu^\sharp_{\mathcal{X}}\mathcal{A}_0$ by the procedure described in the proof of Proposition 4.6. Since

$$\langle S_0,S_{01},\ldots,S_{01\ldots m},S_1,\ldots,S_{1\ldots m}\rangle \;=\; \langle S_1,S_{12},\ldots,S_{1\ldots m},S_0\rangle$$

we deduce that

$$(\mu^\sharp_{S_{1\ldots m}}\cdots\mu^\sharp_{S_1})(\mu^\sharp_{S_{01\ldots m}}\cdots\mu^\sharp_{S_{01}})\mu^\sharp_{S_0}(\mathcal{A}) \;=\; \mu^\sharp_{S_0}(\mu^\sharp_{S_{1\ldots m}}\cdots\mu^\sharp_{S_1})(\mathcal{A})$$

and hence $\mu^\sharp_{\mathcal{F}}(\mathcal{A})=\mu^\flat_{\mathcal{X}[1]}\mu^\sharp_{S_0}\mu^\sharp_{\mathcal{X}}(\mathcal{A})$. $\square$

We use the subsequent lemma as a preparation for Proposition 5.2 below.

Lemma 4.11. *Let $\mathcal{F}_i$ for $i=1,\ldots,r$ be the torsion-free classes lifting the classes $\overline{\mathcal{F}}_i\subset\overline{\mathcal{A}_0}^{(i-1)}$ appearing in the proof of Proposition 4.6 and explicitly described by Lemma 4.10. Then $\mathcal{F}_j\cap\mathcal{F}_i[1]=\{0\}$ for any $j>i$. Moreover, for any $r'\leq r$, the result of the sequence of forward-tilts at $\mathcal{F}_i$ of $\mathcal{A}_0$ for $i=1,\ldots,r'$ is intermediate with respect to $\mathcal{A}_0$.*

Proof. By Lemma 4.10 the class $\mathcal{F}_i$ is generated by a simple object X in $\mathcal{A}^{(i-1)}$ together possibly with extensions of X with $\mathcal{A}^{(i-1)}\cap\mathcal{V}_1$. Similarly is $\mathcal{F}_{i-1}$ for an object Y, with $\pi(X)\neq\pi(Y)$ in $\mathcal{D}/\mathcal{V}_1$, hence $X\neq Y$, and also $X\neq Y[1]$ (since the central charge defines a stability condition on the quotient). None of the extensions of X with $\mathcal{A}_1$ can be in $\mathcal{V}_1$, nor they can just be extensions of $Y[1]$ with $\mathcal{V}_1$. Hence $\mathcal{F}_i\cap\mathcal{F}_{i-1}[1]=\{0\}$ and $\mathcal{F}_{i-1}[1]\subset{}^\perp\mathcal{F}_i$. This is the start for an inductive argument. In fact, we deduce that $\mu^\sharp_{\mathcal{F}_2}\mu^\sharp_{\mathcal{F}_1}\mathcal{A}_0\supset\mathcal{F}_2[1],\mathcal{F}_1[1]$. Now the previous argument shows that $\mathcal{F}_3$ does not intersect $\mathcal{F}_2[1]$ and $\mathcal{F}_1[1]$ non-trivially and we may proceed with the induction.

The last part of the statement then follows from standard facts in tilting theory. $\square$

5. The topology on the space of multi-scale stability conditions

The goal of this section is to provide the space of multi-scale stability conditions $\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})$ with a natural topology so that the quotient by autoequivalences acquires a complex structure (Section 5.3) and so that the further taking the quotient by the $\mathbb{C}$-action gives a compact space (Section 5.4) . The definition of neighborhoods is based on the *plumbing* construction of Section 5.1 and explicitly given in Section 5.2. The name of the construction is derived from [BCGGM3], see also Section 6 where a plumbing construction is performed on Riemann surfaces and where we will see that the constructions are analogous.

5.1. **Plumbing of stability conditions.** The plumbing construction takes as input a multi-scale stability condition $(Z_\bullet, \mathcal{A}_\bullet)$ and a collection $\boldsymbol{\tau} = (\tau_1, \dots, \tau_L) \in -\mathbb{H}^L$ and outputs an honest stability condition. More generally, we will allow $\boldsymbol{\tau}$ to take the formal value $\tau_j = -i\infty$ for j any fixed subset $J \subset \{1, \dots L\}$ and let $-\mathbb{H}_\infty = -\mathbb{H} \cup \{-i\infty\}$. The result of the plumbing construction will then be a multi-scale stability condition with $|J|$ levels below zero, the extreme case $\boldsymbol{\tau} = (-i\infty)^L$ being the identity, no plumbing at all. Using all tuples $\boldsymbol{\tau}$ with each entry of large (or infinite) imaginary part and then allowing small deformations of the resulting generalized stability conditions at each level will provide a neighborhood of $(\mathcal{A}_\bullet, Z_\bullet)$.

For simplicity we start with a multi-scale stability conditions $(\mathcal{A}_\bullet, Z_\bullet)$ with $L = 1$ and let $\mathcal{V} = \mathcal{V}_1^Z$.

Proposition 5.1. *Suppose that the representative $(\mathcal{A}_\bullet, Z_\bullet)$ of a multi-scale stability condition has precisely one level below zero. For $\tau \in -\mathbb{H}$ there is a (honest) stability condition $(\mathcal{A}', Z') = \tau * (\mathcal{A}_\bullet, Z_\bullet)$, the* plumbing of $(\mathcal{A}_\bullet, Z_\bullet)$ with τ, *uniquely determined by the conditions*

- $(\mathcal{A}' \cap \mathcal{V}, Z'|_{K(\mathcal{V})}) = \tau \cdot (\mathcal{A}_1, Z_1)$ *for the usual $\mathbb{C}$-action,*
- *the quotient central charges agree $\overline{Z'} = \overline{Z} \in \mathrm{Hom}(K(\mathcal{D}/\mathcal{V}), \mathbb{C})$, and*
- *the hearts $\overline{\mathcal{A}'} = \overline{\mathcal{A}}_0$ coincide in $\mathcal{D}/\mathcal{V}$.*

Note that the plumbing procedure depends on a chosen representative. For the definition of the topology we will use that the set $\{\tau * (\mathcal{A}_\bullet, Z_\bullet), -\mathrm{Im}(\tau) > C\}$ for any fixed C does not depend on this representative, since the change of representative results in translation of the corresponding τ by a real number. We use that

$$K(\mathcal{A}_0) = K(\overline{\mathcal{A}_0}) \oplus K(\mathcal{A}_1) \quad \text{given by} \quad \mathrm{Sim}(\mathcal{A}_0) = \mathrm{Sim}(\overline{\mathcal{A}_0}) \coprod \mathrm{Sim}(\mathcal{A}_1), \tag{25}$$

(see e.g. the survey [Psa18, Proposition 2.9]) to define two projections

$$\pi_0 : K(\mathcal{D}) \simeq K(\mathcal{A}_0) \to K(\overline{\mathcal{A}_0}), \quad \pi_1 : K(\mathcal{D}) \simeq K(\mathcal{A}_0) \to K(\mathcal{A}_1)\,. \tag{26}$$

Using these projections we combine central charges as

$$Z_0 \oplus Z_1 \;:=\; Z_0 \circ \pi_0 + Z_1 \circ \pi_1\,.$$

Proof. The heart of $\tau \cdot (Z_1, \mathcal{A}_1)$ equals $\mu_{\mathcal{F}} \mathcal{A}_1$ for some torsion-free class $\mathcal{F} \subset \mathcal{A}_1 \subset \mathcal{A}_0$. We take $\mathcal{A}' = \mu_{\mathcal{F}} \mathcal{A}_0$ and by [BMQS22, Lemma 5.6] the last condition holds. Since $\mu_{\mathcal{F}} \mathcal{A}_1 \subset \mathcal{A}'$ and since $\mathcal{A}'$ is a finite heart thanks to $\mathcal{D} = \mathcal{D}^3_{A_n}$ we can use the observation (25) to get the decomposition $K(\mathcal{A}'/\mu_{\mathcal{F}} \mathcal{A}_1) \oplus K(\mu_{\mathcal{F}} \mathcal{A}_1)$ and define $Z = Z_0 \oplus e^{-\pi i \tau} Z_1$. This is indeed a central charge, since $Z(S) \in \overline{\overline{\mathbb{H}}}$ for all simples $S \in \mathcal{A}_1$ by the hypothesis on Z_0 and Z_1. $\square$

Next we generalize to the action of $\boldsymbol{\tau} = (\tau, -i\infty, \ldots, -i\infty)$ on a multi-scale stability condition $(\mathcal{A}_\bullet, Z_\bullet)$. In this case we apply Proposition 5.1 to the first two levels $((\mathcal{A}_0, \mathcal{A}_1), (Z_0, Z_1))$ and record as $\boldsymbol{\tau}$-image the tuple

$$(\tau, -i\infty, \ldots, -i\infty) \cdot (\mathcal{A}_\bullet, Z_\bullet) \;=\; (\mathcal{A}', \mathcal{A}_2, \mathcal{A}_3 \ldots, Z', Z_2, Z_3, \ldots)\,, \tag{27}$$

i.e. the top two levels have been merged to obtain a multi-scale stability condition with $L-1$ levels below zero.

This construction also gives a recipe for the *plumbing* $\boldsymbol{\tau} * (\mathcal{A}_\bullet, Z_\bullet)$ *of a multi-scale stability condition* $(\mathcal{A}_\bullet, Z_\bullet)$ *by a general* $\boldsymbol{\tau} \in -\mathbb{H}^L_\infty$. We take j to be the highest index with $\tau_j \neq -i\infty$. Then we apply the preceding construction to the multi-scale stability condition $(\mathcal{A}_{\geq j}, Z_{\geq j})$ on $\mathcal{V}^Z_j$ and iterate with the action of the remaining coordinates $\boldsymbol{\tau}' = (\tau_1, \cdots, \widehat{\tau_j}, \cdots)$. It will turn out that the plumbing procedure is not quite independent of the order of the levels at which we perform the plumbing step, only nearly so. The reason is that already one-level plumbing and rotation are only nearly compatible. We need a quantitative version of this fact.

Proposition 5.2. *Let* $(\mathcal{A}_\bullet, Z_\bullet)$ *be a fixed representative of a multi-scale stability condition with* $L = 1$. *Let* $\tau \in -\mathbb{H}$, $\lambda \in \mathbb{C}$, *with*

$$0 \leq \operatorname{Re}\lambda, \quad 0 \leq \operatorname{Re}\tau, \quad \textit{and} \quad 0 \leq \operatorname{Re}(\lambda + \tau) < 1. \tag{28}$$

Then the hearts of the two stability conditions

$$\widetilde{\sigma} := (\widetilde{\mathcal{A}}, \widetilde{Z}) := \lambda \cdot (\tau * (\mathcal{A}_\bullet, Z_\bullet)) \quad \textit{and} \quad \widehat{\sigma} := (\widehat{\mathcal{A}}, \widehat{Z}) := \tau * (\lambda \cdot (\mathcal{A}_\bullet, Z_\bullet)) \tag{29}$$

are intermediate hearts with respect to $\mathcal{A}_0$, *and the difference of the central charges may be coarsely estimated by*

$$\Big|(\widehat{Z} - \widetilde{Z})(S_j)\Big| \;\leq\; \ell \cdot |e^{-\pi i(\lambda+\tau)}| \sum_{\substack{S_i \in \mathcal{A}_1 \\ simple}} |Z_1(S_i)| \tag{30}$$

for any simple $S_j \in \operatorname{Sim}(\mathcal{A}_0)$, *where* ℓ *is the number of classes of indecomposables in* $K(\overline{\mathcal{A}_0})$.

We recall from [Bri16, Proposition 7.4] that the local homeomorphism given by the forgetful map $\operatorname{Stab}(\mathcal{D}) \to \operatorname{Hom}(K(\mathcal{D}), \mathbb{C})$ is actually injective when restricted to all hearts that are intermediate with respect to a given heart $\mathcal{A}_0$, i.e., in $[\mathcal{A}_0, \mathcal{A}_0[1]]$. Consequently, to show that $\widetilde{\sigma}$ and $\widehat{\sigma}$ are nearby it suffices to estimate the differences of the central charges if $\widetilde{\mathcal{A}}, \widehat{\mathcal{A}} \in [\mathcal{A}_0, \mathcal{A}_0[1]]$.

Proof. The result of plumbing is given by definition as $(\mathcal{A}, Z) := \tau * (\mathcal{A}_\bullet, Z_\bullet)$ with

$$\mathcal{A} = \mu^\sharp_{\mathcal{F}_\tau}\mathcal{A}_0 \supset \mu^\sharp_{\mathcal{F}_\tau}\mathcal{A}_1, \quad \mathcal{F}_\tau = \langle E \in \mathcal{A}_1,\, Z_1\text{-semistable s.t. } \phi_{Z_1}(E) \leq \operatorname{Re}\tau\rangle$$
$$Z := \overline{Z_0} \oplus e^{-i\pi\tau} \cdot Z_1\,.$$

We first consider the case where

($\star$) there is exactly one isomorphism class $[S_0] \in K(\mathcal{A})$ of a Z-stable object in $\mathcal{A}$ with phase $0 < \phi_Z(S_0) \leq \operatorname{Re}\lambda$, and moreover $[S_0] \notin K(\mathcal{V})$.

(Note that S_0 must be simple in $\mathcal{A}$.) In this case the stability condition $\widetilde{\sigma}$ is given by $\widetilde{Z} = e^{-i\pi\lambda}Z$ and $\widetilde{\mathcal{A}} = \mu^\sharp_{S_0}\mathcal{A} = \mu^\sharp_{\widetilde{\mathcal{F}}}\mathcal{A}_0$ with $\widetilde{\mathcal{F}} = \langle \mathcal{F}_\tau, S_0\rangle$. In particular $\widetilde{\mathcal{A}} \in [\mathcal{A}_0, \mathcal{A}_0[1]]$.

On the other hand the heart $\widehat{\mathcal{A}}$ is obtained by τ-plumbing the multi-scale heart $(\mathcal{A}_1 \subset \mu^\sharp_{\mathcal{F}}\mathcal{A}_0)$, where $\mathcal{F} = \langle S_0, S_{01}, \ldots\rangle \subset \mathcal{A}_0$ is explicitly described in Lemma 4.10, since assumption $(\star)$ implies that the torsion-free class in $\overline{\mathcal{A}_0}$ of objects with phase $0 < \phi_{\overline{Z_0}} \leq \operatorname{Re}\lambda$ is generated by $\overline{S_0}$. Consequently,

$$\widehat{\mathcal{A}} \;=\; \mu^{\sharp}_{\mathcal{G}}\mu^{\sharp}_{\mathcal{F}}\mathcal{A}_0, \quad \text{where} \quad \mathcal{G} \;=\; \langle E \in \mathcal{A}_1,\, Z_1\text{-semistable s.t. } \phi_{Z_1}(E) \leq \operatorname{Re}\tau\rangle$$

Since $\mathcal{G} \subset \mu^{\sharp}_{\mathcal{F}}\mathcal{A}_0 \cap \mathcal{V}_1$ and $\mu^{\sharp}_{\mathcal{F}}\mathcal{A}_1 = \mathcal{A}_1$ we deduce $\mathcal{G} \cap \mathcal{F}[1] = \{0\}$ and consequently (by the same arguments as in Lemma 4.11) we deduce $\mathcal{A}_0 \leq \mu^{\sharp}_{\mathcal{F}}\mathcal{A} \leq \widehat{\mathcal{A}} = \mu^{\sharp}_{\mathcal{G}}\mu^{\sharp}_{\mathcal{F}}\mathcal{A}_0 \leq \mathcal{A}_0[1]$ in the partial order from Section 2.1.

To compare central charges note that the plumbing procedure happens over two different decompositions of $K(\mathcal{D})$: one induced by $K(\mathcal{A}_0/\mathcal{A}_1) \oplus K(\mathcal{A}_1)$, the other induced by $K(\mu^{\sharp}_{\mathcal{F}}\mathcal{A}_0/\mathcal{A}_1) \oplus K(\mathcal{A}_1)$. The change of basis $K(\mu^{\sharp}_{\mathcal{F}}\mathcal{A}_0) \simeq K(\mathcal{A}_0) = K(\mathcal{A}_0/\mathcal{A}_1) \oplus K(\mathcal{A}_1)$ has the form of a block lower-triangular matrix

$$[\mu_{\mathcal{F}}]^{-1} = \begin{pmatrix} C_{n-k} & 0 \\ B_{1\bar{0}} & \mathbf{1}_k \end{pmatrix},$$

where the entries of $B_{1\bar{0}} = (b_{ij})_{ij}$ have absolute value at most 1, where we hardly control the block C_{n-k}, and where the block $\mathbf{1}_k$ expresses that $\mathcal{V}$ is preserved. Using the projections π_0 and π_1 onto the summands $K(\mathcal{A}_0/\mathcal{A}_1) \oplus K(\mathcal{A}_1)$ we find

$$\widetilde{Z} \;=\; e^{-\pi i\lambda}\overline{Z_0} \circ \pi_0 + e^{-\pi i(\lambda+\tau)} Z_1 \circ \pi_1$$

and

$$\begin{aligned} \widehat{Z} \;&=\; e^{-\pi i\lambda} Z_0 \circ \widehat{\pi}_0 + e^{-\pi i(\tau+\lambda)} Z_1 \circ \widehat{\pi}_1 \\ &= e^{-\pi i\lambda} Z_0 \circ \pi_0 + e^{-\pi i(\tau+\lambda)} Z_1 \circ (\pi_1 + B_{1\bar{0}}\pi_0), \end{aligned} \tag{31}$$

where we see $B_{1\bar{0}}$ as a map $K(\mathcal{A}_0/\mathcal{A}_1) \to K(\mathcal{A}_1)$. Now consider the simples in $\mathcal{A}_0$. For $S_j \in \mathcal{A}_1$, the two expressions agree. For S_j a simple of $\mathcal{A}_0$ not in $\mathcal{A}_1$, we find

$$\Big|(\widehat{Z} - \widetilde{Z})(S_j)\Big| \;=\; \Big|\sum_{\substack{S_i \in \mathcal{A}_1 \\ \text{simple}}} (e^{-\pi i(\lambda+\tau)} b_{ij}) Z_1(S_i)\Big| \;\leq\; |e^{-\pi i(\lambda+\tau)}| \cdot \sum_{\substack{S_i \in \mathcal{A}_1 \\ \text{simple}}} |Z_1(S_i)|.$$

We now drop the assumption $(\star)$ and allow for multiple indecomposables $X \in \mathcal{A}$ with $0 < \phi_Z(X) \leq \operatorname{Re}\lambda$.

The assumption (28) still guarantees that $\widetilde{\mathcal{A}} \in [\mathcal{A}_0, \mathcal{A}_0[1]]$, and $\widetilde{Z} = e^{-\pi i\lambda}\overline{Z_0} \circ \pi_0 + e^{-\pi i(\lambda+\tau)} Z_1 \circ \pi_1$, as before.

The result of $\lambda \cdot (\mathcal{A}_\bullet, Z_\bullet)$ is a multi-scale stability condition σ' with nested hearts $\mathcal{A}'_1 \subset \mathcal{A}'_0$ that can be explicitly obtained as in the proof of Proposition 4.6 by performing a forward-tilt at $\mathcal{F}^1_\lambda$ and a sequence of forward-tilts at torsion-free classes $\mathcal{F}_i$ described in Lemma 4.10 and Lemma 4.11. At each step, at the 0 level, the matrix of the change of basis has the form of a block lower triangular matrix, with a block $B_{1\bar{0}} = \prod_i B^{(i)}_{1\bar{0}}$, whose entries have absolute value $\leq \ell'_i$, bounded by the number ℓ'_i of classes of indecomposables in $\mathcal{F}^1_\lambda$ or $\mathcal{F}_i$. Lemma 4.11 guarantees that the heart $\mathcal{A}'_0$ is intermediate with respect to $\mathcal{A}_0$, and so is the heart of $\tau * \sigma'$ thanks to assumption 28. Similarly to (5.1), we obtain

$$\Big|(\widehat{Z} - \widetilde{Z})(S_j)\Big| \leq |e^{-\pi i(\lambda+\tau)}| \cdot \ell \sum_{\substack{S_i \in \mathcal{A}_1 \\ \text{simple}}} |Z_1(S_i)|,$$

where ℓ is the number of classes of indecomposables in $K(\mathcal{A}_0)$. Last, the case $\operatorname{Re}\lambda = 0$ is just easier, and the argument above shows that in such a case $\lambda \cdot (\tau * (\mathcal{A}_\bullet, Z_\bullet)) = \tau * (\lambda \cdot (\mathcal{A}_\bullet, Z_\bullet))$. $\square$

Remark 5.3. *The same observation shows that the plumbing (with fixed parameter $\tau \in -\mathbb{H}$) of a path $\gamma \in \mathrm{MStab}^\circ(\mathcal{D}^3_{A_n}) \setminus \mathrm{Stab}^\circ(\mathcal{D}^3_{A_n})$ is not continuous. Discontinuities occur when some simple (not at bottom level) is tilted. However the size of the jumps decreases with $|e^{-\pi i \tau}|$. More precisely, suppose that $L = 1$ and that γ is a path for which at precisely one value $t_0 \in [0,1]$ such a tilt occurs. Then the hearts of the two stability conditions*

$$\sigma^+ := (\mathcal{A}^+, Z^+) := \lim_{t \to t_0^+} (\tau * \gamma(t)) \quad \text{and} \quad \sigma^- := (\mathcal{A}^-, Z^-) := \lim_{t \to t_0^-} (\tau * \gamma(t))$$

are intermediate hearts with respect to the top level heart $\mathcal{A}_0$ of $\lim_{t \to t_0^-} \gamma(t_0)$ and the difference of the central charges may be coarsely estimated by

$$\left|(Z^+ - Z^-)(S_j)\right| \le \ell \cdot |e^{-\pi i \tau}| \sum_{\substack{S_i \in \mathcal{A}_1 \\ \text{simple}}} |Z_1(S_i)| \tag{32}$$

for any simple $S_j \in \mathrm{Sim}(\mathcal{A}_0)$, where ℓ is the number of classes of indecomposables in $K(\overline{\mathcal{A}_0})$. The proof is exactly the same as for the previous proposition.

Let again $0 \le \mathrm{Re}(\tau) < 1$ and decompose $\tau = \tau_R + i\tau_I$ into its real and imaginary part. We observe that the plumbing in Proposition 4.6 can be viewed a composition of three steps: First we apply the action of τ_R, resulting in a tilt at a torsion-free class $\mathcal{F} \subset \mathcal{A}_1$ and turning Z_1 by $e^{-\pi i \tau_R}$ resulting in some other representative $(\mathcal{A}_\bullet^{\mathrm{Re}(\tau)}, Z_\bullet^{\mathrm{Re}(\tau)})$ of the multi-scale stability condition. Second we rescale $e^{-\pi i \tau_R} Z_1$ by $e^{\pi \tau_I}$, and finally we form the direct sum Z and drop the lower levels to get an honest stability condition. The observation that $[\mu_{\mathcal{F}}]^{-1}$ in the previous proof preserves $\mathcal{V}_1$ and the first step just described does preserve $\mathcal{V}_1$ as well, together imply the following corollary where we relax the bound for $\mathrm{Re}(\tau)$ appearing in Proposition 5.2.

Corollary 5.4. *Let $(\mathcal{A}_\bullet, Z_\bullet)$ be a fixed representative of a multi-scale stability condition with $L = 1$. Let $\tau \in -\mathbb{H}$, $\lambda \in \mathbb{C}$, with $0 \le \mathrm{Re}(\lambda) < 1$. Then the hearts of the two stability conditions*

$$\widetilde{\sigma} := (\widetilde{\mathcal{A}}, \widetilde{Z}) := \lambda \cdot (\tau * (\mathcal{A}_\bullet, Z_\bullet)) \quad \text{and} \quad \widehat{\sigma} := (\widehat{\mathcal{A}}, \widehat{Z}) := \tau * (\lambda \cdot (\mathcal{A}_\bullet, Z_\bullet))$$

are intermediate hearts with respect to $\mathcal{A}_0^{\mathrm{Re}(\tau)}$ and the difference of the central charges may be coarsely estimated as in (30)*.*

Remark 5.5. *For $\lambda \in 2\mathbb{Z}$ and any $\tau \in -\mathbb{H}$ plumbing and the λ-action commute, i.e. the hearts $\widetilde{\sigma}$ and $\widehat{\sigma}$ from* (29) *agree.*

5.2. **Neighborhoods in the space $\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})$.** We work here with a representative $(\mathcal{A}_\bullet, Z_\bullet)$ of a multi-scale stability condition and let

$$\boldsymbol{\delta} = (\delta_1, \ldots, \delta_L)$$

be a tuple of (small) positive real numbers where δ_j will control the size of plumbing of level j. Moreover we fix a collection of positive real numbers

$$\boldsymbol{\varepsilon} = (\varepsilon_J) \quad \text{for any} \quad J \subset \{0, \ldots, L\},$$

e.g., for $J = [01]$, we write ϵ_{01} for ϵ_J. The following Definition 5.6 captures the idea that neighborhoods of boundary points consist of the multi-scale stability conditions, described heuristically as follows. Suppose $L = 1$ for simplicity. Then

- we may plumb by τ with large negative imaginary part (so that the lower level stability condition $(\mathcal{A}_1, e^{-\pi i\tau}Z_1)$ stays small in size) and wiggle the result in $\mathrm{Stab}^\circ(\mathcal{D}^3_{A_n})$ by a small amount (by a size controlled by ε_{01});
- alternatively we may not plumb (i.e. $\tau = -i\infty$) and wiggle in $\mathrm{Stab}^\circ(\mathcal{V}_1)$ and $\mathrm{Stab}^\circ(\mathcal{D}/\mathcal{V}_1)$ a bit (by sizes controlled by ε_i) on level i, for $i = 0, 1$.

We say that a stability condition $(\mathcal{A}'_\bullet, Z'_\bullet)$ with $L' \le L$ levels below zero arises *by plumbing of size at most* $\boldsymbol{\delta}$ *from* $(\mathcal{A}_\bullet, Z_\bullet)$ if there is $\tau \in (-\mathbb{H}_\infty)^L)$ with $|e^{-\pi i\tau_j}| < \delta_j$ for $j = 1, \ldots, L$ and $(\mathcal{A}'_\bullet, Z'_\bullet) = \tau * (\mathcal{A}_\bullet, Z_\bullet)$. For a such a stability condition we denote the new vanishing categories by $\mathcal{V}'_i$. Say levels in the interval $J_i = \{j_1, \ldots\} \subset \{0, \ldots, L\}$ have been plumbed to form the new level i. (This implies by definition that $\tau_{j_1} = -i\infty$.)

We define the *natural inner product* on $K(\mathcal{V}'_i/\mathcal{V}'_{i+1})^\vee = \mathrm{Hom}(K(\mathcal{V}'_i/\mathcal{V}'_{i+1}), \mathbb{C})$ by using as an orthonormal basis the basis Z_{S_i} dual to the simples of $\mathcal{A}_0$. (Note that this norm depends on the heart $\mathcal{A}_0$ and its simples, but the norm around any other multi-scale stability condition $\sigma^\dagger = (\mathcal{A}^\dagger_\bullet, Z^\dagger_\bullet)$ is comparable, scaling by a factor $C = C(\sigma, \sigma^\dagger)$ given by the operator norm of the identity map with respect to the to norms.)

Definition 5.6. *We define the set $V_{\boldsymbol{\varepsilon},\boldsymbol{\delta}}(\mathcal{A}_\bullet, Z_\bullet)$ to be the set of all multi-scale stability conditions $(\mathcal{A}''_\bullet, Z''_\bullet)$ with L' levels below zero such that*

(1) there is a multi-scale stability condition $(\mathcal{A}'_\bullet, Z'_\bullet)$ with $L' \le L$ levels that arises by plumbing of size at most $\boldsymbol{\delta}$ from $(\mathcal{A}_\bullet, Z_\bullet)$, and

(2) the multi-scale stability condition $(\mathcal{A}''_\bullet, Z''_\bullet)$ is in a neighborhood of $(\mathcal{A}'_\bullet, Z'_\bullet)$ in $\prod \mathrm{Stab}^\circ(\mathcal{V}'_i/\mathcal{V}'_{i+1})$ which maps to the product of ε_J-balls on $K(\mathcal{V}'_i/\mathcal{V}'_{i+1})^\vee$ under the forgetful map retaining just the quotients of the multi-scale central charges. Here J is the interval that is plumbed to produce level i.

A neighborhood *of $(\mathcal{A}_\bullet, Z_\bullet,)$ is a set in $\mathrm{MStab}^\circ(\mathcal{D}_{A_n})$ that contains $V_{\boldsymbol{\varepsilon},\boldsymbol{\delta}}(\mathcal{A}_\bullet, Z_\bullet,)$ for some $\boldsymbol{\varepsilon}$ and $\boldsymbol{\delta}$.*

This definition includes the case that $L = 0$ and that $(Z, \mathcal{A})$ is an honest stability condition, in which case the neighborhoods have to contain the ε-balls in the norm with orthonormal bases given by the simples of $\mathcal{A}$, since the deformation of stability conditions is locally controlled by the deformation of the central charge. This gives the second part of the following lemma.

Lemma 5.7. *The system of neighborhoods given in Definition 5.6 defines a topology on $\mathrm{MStab}(\mathcal{D}^3_{A_n})$ whose restriction to $\mathrm{Stab}(\mathcal{D}^3_{A_n})$ is the usual topology where the forgetful map retaining the central charge is a local homeomorphism.*

Proof. The only axiom whose verification is non-trivial is the following. Let U be a neighborhood of $\sigma = [A_\bullet, Z_\bullet]$, in the sense of Definition 5.6. Then there is a smaller neighborhood V of this point, such that U is a neighborhood of each $\sigma^\dagger = [\mathcal{A}^\dagger_\bullet, Z^\dagger_\bullet]$ in V. We continue with the case $L = 1$, the general case works with the same argument. By definition U contains some $V_{\boldsymbol{\varepsilon},\delta}(A_\bullet, Z_\bullet)$. The rough idea is to take $V = V_{\boldsymbol{\varepsilon}^*,\delta^*}(A_\bullet, Z_\bullet)$ for some $(\boldsymbol{\varepsilon}^*, \delta^*)$ smaller than $(\boldsymbol{\varepsilon}, \delta)$ in each entry, so that U contains $V_{(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^*)/C,\delta-\delta^*}(\sigma^\dagger)$, just as if we'd be working plainly in vector spaces, where $C = C(\sigma, \sigma^\dagger) \ge 1$ accounts for the change of basis in the definition of the norms. We will prove that $V_{(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^*)/C,\delta-\delta^*}(\sigma^\dagger)$ is indeed contained in $V_{\boldsymbol{\varepsilon},\delta}(\sigma)$ for $\boldsymbol{\varepsilon}^*$ carefully chosen. We have to avoid that ε^*_0, ε^*_1 are large compared to ε^*_{01} so that any plumbing after deforming $(\mathcal{A}^\dagger_\bullet, Z^\dagger_\bullet)$ of the order of ε_0, ε_1 doesn't fail of being near σ.

We may thus first take the pair $(\varepsilon_0^*, \varepsilon_1^*)$ so small that any point in the $(\varepsilon_0^*, \varepsilon_1^*)$-ball in $\mathrm{Stab}^\circ(\mathcal{D}/\mathcal{V}_1) \times \mathrm{Stab}^\circ(\mathcal{V}_1)$ can be reached from σ by a path $\gamma(t)$ involving at most one tilt, at $t = 0$. Suppose the chosen $\sigma^\dagger = (A_\bullet^\dagger, Z_\bullet^\dagger) \in V_{\boldsymbol{\varepsilon}^*, \delta^*}(\sigma)$ has also $L = 1$ levels below zero and let $\gamma^\dagger(t)$ be the straight path connecting σ and $\sigma^\dagger$. Let τ in $-\mathbb{H}$ of magnitude at most δ^*, and $\hat{\sigma} = \tau * \sigma^\dagger$, $\sigma' = \tau * \sigma$. By the argument in Proposition 5.2 and the one-tilt hypothesis, the hearts $\hat{\mathcal{A}}$ and $\mathcal{A}'$ are intermediate with respect to $\mathcal{A}_0$ so the distance between $\hat{\sigma}$ and σ' is controlled by their central charges. If the plumbing of the path $\gamma^\dagger$ is continuous, then

$$||\widehat{Z} - Z'|| \le \varepsilon_0^* + \delta_1^* \varepsilon_1^* \,. \tag{33}$$

In the general case of a single tilt use Remark 5.3 and compare $\sigma' = \sigma^-$ with $\sigma^+ := (\mathcal{A}^+, Z^+) = \lim_{t\to 0^+} \tau * \gamma^\dagger(t)$. Now (32) and the previous estimate in the new notation give the rough estimate

$$||\widehat{Z} - Z^+|| \le \varepsilon_0^* + \delta_1^* \varepsilon_1^* \quad \text{and} \quad ||Z^+ - Z'|| \le \ell n \delta_1^*. \tag{34}$$

The triangle inequality shows that requiring moreover $\varepsilon_{01}^* \le \varepsilon_0^* + \delta_1^*(\varepsilon_1^* + \ell n)$ does the job. $\square$

The next lemma will be used in the proofs at the end of this section. We consider a sequence $\{\sigma_j\}_j$ of multi-scale stability conditions in $\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})$.

Lemma 5.8. *Fix $\lambda \in \mathbb{C}$ with $0 < \mathrm{Re}(\lambda) < 1$ and consider the $\mathbb{C}$-action given in Proposition 4.6. Then a sequence $\{\sigma_j\}_j$ converges to σ in $\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})$ if and only if $\{\lambda \cdot \sigma_j\}_j$ converges to $\lambda \cdot \sigma$.*

Proof. Again we give the details in the case that σ has $L = 1$ levels below zero. Extracting subsequences we may assume that all σ_j have the same number of levels below zero. If they are strict multi-scale stability conditions, the claim follows from the corresponding statement in $\mathrm{Stab}^\circ(\mathcal{V}_1) \times \mathrm{Stab}^\circ(\mathcal{D}^3_{A_n}/\mathcal{V}_1)$. The interesting case is that $\sigma_j \in \mathrm{Stab}^\circ(D^3_{A_n})$ for all j.

Convergence and the definition of neighborhoods implies that σ_j is in a open set $V_{\varepsilon_j, \delta_j}(\sigma_j')$, where $\sigma_j' = \tau_j * \sigma$ for some fixed representative of σ and for both $\varepsilon_j \to 0$ and $\delta_j = |e^{-\pi i \tau_j}| \to 0$ as $j \to \infty$. We apply the action of λ to this sequence and use Corollary 5.4 to see that $\lambda \cdot (\tau_j * \sigma)$ is close to $\tau_j * (\lambda \cdot \sigma)$ in a way controlled by (30). We conclude that $\lambda \cdot \sigma_j$ is in an ε_j'-ball of $\lambda \cdot (\tau_j * \sigma)$ for $\varepsilon_j' = \varepsilon_j + \ell \delta_j$, which certifies convergence. $\square$

The Hausdorff property. We now start proving that $\mathrm{MStab}^\circ(D^3_{A_n})$ is a nice topological space.

Lemma 5.9. *The space $\mathrm{MStab}^\circ(D^3_{A_n})$ is second countable.*

Proof. The basis of neighborhoods consisting of $V_{\boldsymbol{\varepsilon}, \boldsymbol{\delta}}(\mathcal{A}_\bullet, Z_\bullet)$ with $(\mathcal{A}_\bullet, Z_\bullet)$ such that the Z_i map the collection of simples to a (projectivized) tuple of rational numbers, and with all entries of $(\boldsymbol{\varepsilon}, \boldsymbol{\delta})$ being rational obviously generates the same topology as the one using real numbers. $\square$

Theorem 5.10. *For any subgroup $G \subset \mathrm{Aut}^\circ(\mathcal{D}^3_{A_n})$ the quotient $\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})/G$ and the projectivized version $\mathbb{P}\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})/G$ are Hausdorff topological spaces.*

Proof. Since the relevant (quotient) spaces are second countable, being Hausdorff is equivalent to uniqueness of limits, which we now show. Suppose that the sequence $\sigma_j = [\mathcal{A}_{\bullet,j}, Z_{\bullet,j}]$ of multi-scale stability condition converges to $\sigma = [\mathcal{A}_\bullet, Z_\bullet]$ and that the sequence $\sigma'_j = \Phi_j[\mathcal{A}_{\bullet,j}, Z_{\bullet,j}]$, with $\Phi_j \in G$, converges to $\sigma' = [\mathcal{A}'_\bullet, Z'_\bullet]$ in $\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})$. We need to show that $[\mathcal{A}'_\bullet, Z'_\bullet] = \Phi[\mathcal{A}_\bullet, Z_\bullet]$ for some $\Phi \in G$. We restrict our argument to the cases that all the σ_j are honest stability conditions, that $[\mathcal{A}_\bullet, Z_\bullet]$ has $L = 1$ level below zero and that $[\mathcal{A}'_\bullet, Z'_\bullet]$ has $L' \in \{0, 1\}$, leaving the inductive arguments to treat the general case to the reader. Note that the mass $M_{\max}(\sigma_j)$ of the longest and the mass $M_{\min}(\sigma_j)$ of the shortest stable object in σ_j is a notion that is invariant under the action of $\mathrm{Aut}^\circ(\mathcal{D}^3_{A_n})$.

The case $L' = 0$ is absurd, since this implies that $M_{\min}(\sigma_j)/M_{\max}(\sigma_j)$ is bounded below, while the convergence to σ implies that this ratio tends to zero.

In general, for a sequence σ_j converging to σ with $L = 1$ and for some cut-off parameter $M > 1$, we say that a simple S is 'short' if its mass is less that $1/M$ times the largest mass of a simple, and 'long' otherwise.

In the case $L' = 1$ consider the set of short stable objects in the sequences σ_j and σ'_j respectively. By definition of the topology, these short stable objects eventually (as $C \to \infty$) generate the vanishing subcategories $\mathcal{V}$ and $\mathcal{V}'$. Consequently, $\Phi_j \mathcal{V} = \mathcal{V}'$ for $j \geq N$ for some N large enough. Replacing Φ_j by $\Phi_N^{-1} \circ \Phi_j$ we may suppose from now on that $\mathcal{V} = \mathcal{V}'$ and $\Phi_j \in G \cap \mathrm{Aut}^\circ(\mathcal{D}^3_{A_n}, \mathcal{V})$ for all $j \geq N$. Using the triangle inequality and the definition of the metric, it is easy to show that also the sequence $\Phi_j(\sigma)$ converges to σ' in $\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})$. Since all these objects are now in fact in $\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n}, \mathcal{V})$, this implies that $\Phi_j(\mathcal{A}_0, Z_0) \to (\mathcal{A}'_0, Z'_0)$ in $\mathrm{Stab}^\circ(\mathcal{D}^3_{A_n}/\mathcal{V})$ and $\Phi_j[\mathcal{A}_1, Z_1] \to [\mathcal{A}'_1, Z'_1]$ as projectivized stability conditions in $\mathbb{P}\,\mathrm{Stab}^\circ(\mathcal{V})$. Since $\mathrm{Aut}^\circ(\mathcal{D}^3_{A_n}, \mathcal{V})$ acts on $\mathrm{Stab}^\circ(\mathcal{D}^3_{A_n}/\mathcal{V})$ via $\mathcal{A}ut_{\mathrm{lift}}(\mathcal{D}/\mathcal{V})$, by Lemma 4.3 the image $\mathrm{Im}(G) \subset \mathrm{Aut}^\circ(\mathcal{D}^3_{A_n}/\mathcal{V})$ acts properly discontinuously on $\mathrm{Stab}^\circ(\mathcal{D}^3_{A_n}/\mathcal{V})$. By definition the stabilizer in a neighborhood of $(\overline{\mathcal{A}_0}, \overline{Z_0})$ is finite, hence after passing to a sub-sequence, we may assume (again for $j \geq N$, which we assume now throughout) that $\overline{\Phi}_j \equiv \overline{\Phi}^{(N)} \in \mathrm{Im}(G) \subset \mathrm{Aut}^\circ(\mathcal{D}^3_{A_n}/\mathcal{V})$ with $\overline{\Phi}^{(N)}(\overline{\mathcal{A}_0}, \overline{Z_0}) = (\overline{\mathcal{A}'_0}, \overline{Z'_0})$.

On lower level the stabilizer of $[\mathcal{A}_1, Z_1]$ is not finite. Let H be the stabilizer of the un-projectivized $(\mathcal{A}_1, Z_1)$. We recall that, by Lemma 4.3 the group H is a finite extension of $Z(B_{\mathrm{rank}(K(\mathcal{V}))})$, which in turn acts trivially on the projectivized $[\mathcal{A}_1, Z_1]$. This implies that, at the cost of passing to a sub-sequence, there are $z_j \in Z(B_{\mathrm{rank}(K(\mathcal{V}))})$ such that $\overline{\Phi}_j\big(z_j[\mathcal{A}_1, Z_1]\big)$ is in fact the same converging sequence, but we can now work with their representatives in $\mathrm{Stab}^\circ(\mathcal{D}^3_{A_n}/\mathcal{V})$ that have finite stabilizer. We can extract a convergent subsequence with $\Phi_{j|\mathcal{V}}(\mathcal{A}_1, Z_1) \equiv (\mathcal{A}'_1, Z'_1)$. Taken together, this means that there is N large enough so that $\Phi_N \sigma_\bullet = \sigma'_\bullet$.

The case $\mathbb{P}\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})$ is similar: one then has to correct also the top level by a central element to ensure convergence to some $\overline{\Phi^{(0)}}$. □

5.3. **The complex structure on quotients of boundary neighborhoods in $\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})$.** Next we upgrade from a topology to a structure of complex orbifold. This will not be possible on $\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})$, but only the quotient by the group of autoequivalences, see Section 6.4 for an illustration in the case of the A_2-quiver. The first step in this direction is to exhibit a subgroup $\mathrm{Tw}^s(\mathcal{V}_\bullet)$ in the stabilizer of the boundary such that the quotient $\mathrm{MStab}^\circ(\mathcal{D}, \mathcal{V}_\bullet)/\,\mathrm{Tw}^s(\mathcal{V}_\bullet)$ is a complex manifold.

We then determine the full stabilizer of these boundary neighborhoods and exhibit the orbifold structure.

We fix once for all a multi-scale stability condition $\sigma_\bullet = [Z_\bullet, \mathcal{A}_\bullet]$ with L levels below zero, and let $\mathcal{V}_\bullet$ be the associated sequence of nested vanishing subcategories of $\mathcal{D} = \mathcal{D}^3_{A_n}$.

The simple twist group $\mathrm{Tw}^s(\mathcal{V}_\bullet)$. Recall the numerical data associated with a multi-scaled stability conditions of type A_n from Section 4.1 and suppose $\sigma_\bullet$ is of type $\boldsymbol{\rho}$. We focus at a level i and let $\mathcal{V}_i^{(j)}$ be the components of $\mathcal{V}_i$. Suppose that $\mathcal{V}_i^{(j)}$ has type $n_i^{(j)}$, i.e. the heart $\mathcal{V}_i^{(j)} \cap \mathcal{A}_i$ has $n_i^{(j)}$ simples $S_1, \ldots, S_{n_i^{(j)}}$ in the subset $\mathcal{S} = \mathcal{S}(i,j)$ of $\mathrm{Sim}(\mathcal{A}_0)$. We recall the definition in Section 3.3 of the group elements $\theta_{I,n}$.

For I denoting the closed arcs in the subsurface $\Sigma_i^{(j)}$ we let

$$\mathfrak{c}_{i,j} = \begin{cases} \theta_{I,n} & \text{if } n_i^{(j)} \text{ is odd} \\ \theta_{I,n}^2 & \text{if } n_i^{(j)} \text{ is even.} \end{cases} \tag{35}$$

From $\boldsymbol{\rho}$ we derive another collection of integers $(\ell_i)_{i=1}^L$. Recall that we define $\kappa_i^{(j)} = n_i^{(j)} + 3$ to be the number of marked points on the boundary of $\Sigma_i^{(j)}$. We let

$$\widehat{\kappa}_i^{(j)} \;=\; \begin{cases} (n_i^{(j)} + 3)/2 & \text{if } n_i^{(j)} \text{ is odd} \\ (n_i^{(j)} + 3) & \text{if } n_i^{(j)} \text{ is even} \end{cases} \tag{36}$$

(This notation is consistent with the enhancements in Section 6.) We define

$$\ell_i \;=\; \mathrm{lcm}\{\widehat{\kappa}_i^{(j)}\,,\, j = 1, \ldots, s_i\}\,. \tag{37}$$

For each level i and each j we now define the elements

$$\mathfrak{c}_i := \prod_j \mathfrak{c}_{i,j}^{\ell_i/\widehat{\kappa}_i^{(j)}} := \prod_j \theta_{I(i,j),n}^{\ell_i/\kappa_i^{(j)}}. \tag{38}$$

and we define the *simple twist group* to be $\mathrm{Tw}^s(\mathcal{V}_\bullet) := \langle \mathfrak{c}_1, \ldots, \mathfrak{c}_L \rangle$.

Proposition 5.11. *For each level i the element $\mathfrak{c}_i \in \mathrm{PB}_n$ preserves the neighborhoods $V_{\varepsilon,\delta}(\mathcal{A}_\bullet, Z_\bullet)$ for all (ε, δ) small enough.*

Implicit in the notation is that the elements $\mathfrak{c}_{i,j}$ for fixed j commute. In fact:

Lemma 5.12. *The elements $\mathfrak{c}_{i,j}$ for all (i,j) commute. In particular the* simple twist group $\mathrm{Tw}^s(\mathcal{V}_\bullet) = \langle \mathfrak{c}_1, \ldots, \mathfrak{c}_L \rangle$ *is a free abelian group of rank L.*

Proof. For fixed i the elements $\mathfrak{c}_{i,j}$ and $\mathfrak{c}_{i,j'}$ commute by (14) since they correspond to disjoint subsurfaces and hence any two vertices, one in $I(i,j)$ and one in $I(i,j')$, cannot be connected by an edge in the corresponding quiver. For different level indices $i < i'$, the elements $\mathfrak{c}_{i,j}$ and $\mathfrak{c}_{i',j'}$ commute for the same reason if $Q_{I(i',j')}$ is not a subquiver of $Q_{I(i,j)}$. If it is a subquiver, the elements commute since $\theta_{I,n}$ is (the image of) the central element in the braid group corresponding to $Q_{I(i,j)}$. $\square$

Proof of Proposition 5.11. Focusing on the subcategories above and below i we may reduce to the case that $(\mathcal{A}_\bullet, Z_\bullet)$ has $L = 1$ and we consider $i = 1$, thus writing

$\mathcal{V} = \mathcal{V}_1$. Suppose the connected components $\mathcal{V}^{(j)}$ have type n_j and correspond to the subsurfaces $\Sigma^{(j)}$. We claim that

$$\begin{aligned}
\mathfrak{c}_{1,j}(\mathcal{A}_1 \cap \mathcal{V}^{(j)} Z_1|_{K(\mathcal{V}^{(j)})}) &= \widehat{\kappa}_1^{(j)} \cdot (\mathcal{A}_1 \cap \mathcal{V}^{(j)}, Z_1|_{K(\mathcal{V}^{(j)})}), \\
\mathfrak{c}_{1,j} Z_0 &= Z_0 \in \operatorname{Hom}(K(\mathcal{D}^3_{A_n}/\mathcal{V}), \mathbb{C}), \\
\mathfrak{c}_{1,j}\overline{\mathcal{A}}_0 &= \overline{\mathcal{A}}_0 \in \mathcal{D}^3_{A_n}/\mathcal{V}\,.
\end{aligned} \tag{39}$$

Granting the claim, we conclude that $\mathfrak{c}_1$ acts like the shift by ℓ_1 on $\mathcal{A}_1 \cap \mathcal{V}^{(j)}$ for every j, thus on the whole $(\mathcal{A}_1, Z_1)$. Since adding $\widehat{\kappa}_1^{(j)}$ to τ does not change the norm used in (1) of the topology definition, the elements stabilize the neighborhoods as claimed. Since $\tau + \widehat{\kappa}_1^{(j)}$ realizes a plumbing of size at most δ, if τ does so, the elements stabilize the neighborhoods (as defined by (1)-(2) in Definition 5.6) as claimed.

The first equality of (39) is [ST01, Lemma 4.14] applied to the subquiver $Q_{I(1,j)}$. The second equality holds by definition of the induced actions on Grothendieck groups, see (19). For the third equality we write each spherical twist appearing in the definition of $\mathfrak{c}_{1,j}$ as a composition of tilts at simples in $\mathcal{V}$ using (16). The claim then follows since such tilts do not change the quotient heart, see [BMQS22, Lemma 5.6]. □

The complex structure on boundary neighborhoods. Consider a neighborhood $V_{\varepsilon,\delta}(\mathcal{A}_\bullet, Z_\bullet)$ as given in Definition 5.6. As we stated after (39) the element $\mathfrak{c}_i$ acts as the shift by ℓ_i on the i-th level of $(\mathcal{A}_\bullet, Z_\bullet)$. Consequently, plumbing by $\boldsymbol{\tau}$ and plumbing by $\boldsymbol{\tau} + \ell_i e_i$ (where e_i is the i-th unit vector) give the same stability condition in the quotient $V_{\varepsilon,\delta}(\mathcal{A}_\bullet, Z_\bullet)/\operatorname{Tw}^s(\mathcal{V}_\bullet)$. Therefore, on this quotient space the parameters

- $t_i = \exp\{2\pi\sqrt{-1}\frac{\tau_i}{\ell_i}\}$ for $i = 1, \ldots, L$,
- the ratios of central charges of simples on $\mathbb{P}\operatorname{Stab}^\circ(\mathcal{V}_j/\mathcal{V}_{j+1})$ for $j > 0$,
- the central charges of the simples on $\operatorname{Stab}^\circ(\mathcal{D}^3_{A_n}/\mathcal{V}_1)$,

all together give a complex chart.

Instead of using ratios of central charges we may equivalently fix a representative $(\mathcal{A}_\bullet, Z_\bullet)$ of the multi-scale stability condition such that a ('pivot') simple S_i in each $\mathcal{A}_i$ for $i > 0$ has $Z(S_i) = 1$ and use central charges of the remaining (non-pivot) simples in $\operatorname{Sim}(\mathcal{A}_i) \setminus \operatorname{Sim}(\mathcal{A}_{i+1})$ together with the t_i as coordinates.

Next we check compatibility, i.e., we compare two charts defined around points in $V_{\varepsilon,\delta}(\mathcal{A}_\bullet, Z_\bullet)/\operatorname{Tw}^s(\mathcal{V}_\bullet)$ and $V'_{\varepsilon',\delta'}(\mathcal{A}'_\bullet, Z'_\bullet)/\operatorname{Tw}^s(\mathcal{V}'_\bullet)$ with $L \neq L'$, in the case of non-trivial intersection of the neighborhoods. In particular we check compatibility with the existing complex structure on $\operatorname{Stab}^\circ(\mathcal{D}^3_{A_n})$. Fix $\sigma = (\mathcal{A}_\bullet, Z_\bullet)$ with $L = 1$ and consider a point $\sigma' = (\mathcal{A}'_\bullet, Z'_\bullet) \in V_{\varepsilon,\delta}(\mathcal{A}_\bullet, Z_\bullet)$ and suppose first that $L = 1$ and $\sigma' = (Z, \mathcal{A})$ is actually an honest stability condition. We now use the definition of the plumbed central charge according to Proposition 5.1 and the 'pivot' viewpoint for coordinates. We find that the central charge $Z'_0(S)$ of a simple $S \in \operatorname{Sim}(\mathcal{A})$ is equal to $Z_0(S)$ if $S \notin \operatorname{Sim}(\mathcal{A}_1)$ or equal to $t_1 Z_1(S)$ if $S \in \operatorname{Sim}(\mathcal{A}_i)$. Since $t_1 \neq 0$ near σ', this coordinate change is a biholomorphism. Similarly, the compatibility holds if $L > 1$ and if the neighboring point σ' has $L' > 0$ below zero, the coordinate change map being given by a product of t_j's times $Z_i(S)$ for $S \in \operatorname{Sim}(\mathcal{A}_i) \setminus \operatorname{Sim}(\mathcal{A}_{i+1})$. We summarize this discussion:

Proposition 5.13. *The quotients of the boundary neighborhoods $V_{\varepsilon,\delta}(\mathcal{A}_\bullet, Z_\bullet)$ by the simple twist group $\mathrm{Tw}^s(\mathcal{V}_\bullet)$ admit a complex structure compatible with the complex structure around any point $(\mathcal{A}'_\bullet, Z'_\bullet) \in V_{\varepsilon,\delta}(\mathcal{A}_\bullet, Z_\bullet)$ with less that L levels below zero.*

The orbifold structure. The following proposition ensures that the complex structure on $\mathrm{Tw}^s(\mathcal{V}_\bullet)$-quotients of neighborhoods actually gives the structure of an orbifold on the quotient space $\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})/\mathcal{A}ut(\mathcal{D}^3_{A_n})$.

Proposition 5.14. *The stabilizer in $\mathcal{A}ut(\mathcal{D}^3_{A_n})$ of a multi-scale stability condition $\sigma_\bullet = (\mathcal{A}_\bullet, Z_\bullet)$ contains $\mathrm{Tw}^s(\mathcal{V}_\bullet)$ as a finite index subgroup.*

Proof. For the intended finiteness assertion we may restrict attention from $\mathcal{A}ut(\mathcal{D}^3_{A_n})$ to the group of spherical twists $\mathrm{ST}(A_n) \cong B_{n+1}$ thanks to (18).

Suppose that $L = 1$ and suppose moreover that the lower level is connected. Consider the intersection G of the stabilizer $H_{\sigma_\bullet}$ with $\mathrm{ST}(A_n)$. We know that any $\rho \in G$ stabilizes $\mathcal{V}$ and fixes the (lower level) stability condition on $\mathcal{V}$ projectively. This implies as in the proof of Lemma 4.3 that after passing to a finite index subgroup of $G = H_{\sigma_\bullet} \cap \mathrm{ST}(A_n)$ we may assume that $\rho|_{\mathcal{V}}$ as an element of $\mathrm{Aut}^\circ(\mathcal{V})$ is central, i.e., a power of $\theta_{I,n}$, since the $\mathbb{C}$-action and the action of autoequivalences commute. Consequently $\langle \mathfrak{c}_1 \rangle = \mathrm{Tw}^s(\mathcal{V}_\bullet) \subset G$ of finite index.

Suppose still $L = 1$ but now that the lower level has say k connected components $\mathcal{V}^{(j)}$. Now the preceding argument implies that after passing to a finite index subgroup G of the stabilizer (thereby getting rid of potential non-trivial pointwise stabilizers of $(Z_1, \mathcal{V}_1)$) any $\rho|_{\mathcal{V}^{(j)}}$ for $\rho \in G$ is central in the autoequivalence group of each component of $\mathcal{V}^{(j)}$, i.e. a power of the $\mathfrak{c}_{1,j}$. We now use that moreover the elements in G act by simultaneously rescaling the restrictions of $(Z_1, \mathcal{V}_1)$ to the components $\mathcal{V}^{(j)}$ by definition of the equivalence relation of multi-scale stability condition. We deduce that G is a cyclic group. Since the exponents in the definition of $\mathfrak{c}_1$ were chosen to projectivize simultaneously (raise the first equation of (39) to the right power), the claim follows in this case.

For $L > 1$ a new phenomenon occurs: Suppose $\mathcal{A}_i \cap \mathcal{V}_i^{(j)} = \mathcal{A}_{i+1} \cap \mathcal{V}_{i+1}^{(j')}$ for certain components $\mathcal{V}_i^{(j)}$ and $\mathcal{V}_{i+1}^{(j')}$ of the vanishing subcategories at level i and $i+1$. This is possible if $Z_i|_{\mathcal{A}_i \cap \mathcal{V}_i^{(j)}} = 0$ and the required non-vanishing of Z_i is ensured on some other component $\mathcal{V}_i^{(k)}$ of $\mathcal{V}$. Then the condition 'projectively equivalent' in the definition of a multi-scale stability condition imposes no constraint relating the action on $(\mathcal{A}_i \cap \mathcal{V}_i^{(j)}, Z_i|_{\mathcal{V}_i^{(j)}})$ and $(\mathcal{A}_i \cap \mathcal{V}_i^{(k)}, Z_i|_{\mathcal{V}_i^{(k)}})$. We capture this problem as follows:

We write G for the finite index subgroup of the stabilizer that acts trivially on each $(\mathcal{V}_i, Z_i)$. Let E be the set of (homotopy classes of) seams of the subsurfaces corresponding to the components $\mathcal{V}_i^{(j)}$ and identify $\mathbb{Z}^E$ with the group generated by the $\theta_{I(i,j),n}$, i.e., the group generated by the twist around these seams. Then there is a natural embedding $G \to \mathbb{Z}^E$. The image is contained for each level i by $C_i - 1$ constraints due to simultaneous projectivization, where C_i is the number of components where $\mathcal{A}_i \cap \mathcal{V}_i^{(j)} \neq \mathcal{A}_{i+1} \cap \mathcal{V}_{i+1}^{(j')}$, i.e., where the seam of the subsurface at level i does not agree with the seam of the subsurface at level $i+1$. Since $\sum_{i=1}^{L} C_i = E$ and since these constraints are obviously independent we conclude that G is a free group of rank L. Since $\mathrm{Tw}^s(\mathcal{V}_\bullet) \subset G$ is also free group of rank L, this must be an inclusion of finite index. □

The group G appearing in the last paragraph of the proof should be called the full twist group $\mathrm{Tw}(\mathcal{V}_\bullet)$ in analogy with the full twist group of level graphs appearing in [BCGGM3, Section 6], see also Section 6. The factor group $\mathrm{Tw}(\mathcal{V}_\bullet)/\mathrm{Tw}^s(\mathcal{V}_\bullet)$ is thus responsible for the orbifold structure of $\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})/\mathrm{Aut}(\mathcal{D}^3_{A_n})$.

5.4. **Compactness.** Our goal is:

Theorem 5.15. *For any finite index subgroup $G \subset \mathrm{Aut}(\mathcal{D}^3_{A_n})$ the quotient space $\mathbb{P}\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})/G$ is compact.*

Proof. Since $\mathbb{P}\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})/G$ is second countable, because G is countable, compactness is equivalent to being sequentially compact. Given a sequence σ_m of stability conditions we want to extract a convergent sub-sequence after rescaling the family appropriately. Since the quotient space $\mathrm{Stab}^\circ(\mathcal{D}^3_{A_n})/G$ has finitely many chambers (given by undecorated triangulations of the disc) and since the stability spaces of quotient categories involved in $\mathbb{P}\mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})/G$ have the same property (corresponding to partial triangulations, see [BMQS22]) we may modify σ_m by suitable elements of G and pass to a subsequence and assume that all elements of σ_m belongs to a single chamber. We will moreover assume that $\sigma_m = [\mathcal{A}, Z^{(m)}] \in \mathbb{P}\,\mathrm{Stab}(\mathcal{A}) \subset \mathbb{P}\,\mathrm{Stab}^\circ(\mathcal{D}^3_{A_n})$ are honest stability conditions. At the end of the proof it will be clear that the general case follows by the same argument, just using an extra index for the levels of the initial multi-scale stability conditions.

We define a (weak) full order $\succcurlyeq$ on $\mathrm{Sim}(\mathcal{A})$ by

$$S_1 \succcurlyeq S_2 \quad \text{if} \quad \inf_{m\in\mathbb{N}} |Z^{(m)}(S_1)|/|Z^{(m)}(S_2)| > 0\,,$$

Equivalently, S_1 is strictly smaller than S_2 if the ratio of central charges tends to zero. Since there are finitely many simples, we may index the level sets of this order by integers $0, 1, \ldots, L$ and use these to generate Serre subcategories of $\mathcal{A}$. For consistence of indexing we assume that $\mathcal{A}_L$ is generated by the set of smallest simples (with respect to $\succcurlyeq$), that $\mathcal{A}_{L-1}$ is generated by $\mathrm{Sim}(A_L)$ and the set of second smallest simples, etc., thus arriving at a nested sequence

$$\mathcal{A}_L \subset \mathcal{A}_{L-1} \subset \cdots \subset \mathcal{A}_1 \subset \mathcal{A}_0 = \mathcal{A}\,.$$

Let's order the simples of $\mathcal{A}$ so that $(S_1, \ldots, S_{r_0}) \notin \mathrm{Sim}(\mathcal{A}_1)$, using implicitly the definition $r_0 = \mathrm{rank}(K(\mathcal{A}_0)) - \mathrm{rank}(K(\mathcal{A}_1)) \geq 1$. Since $\mathbb{P}^{n-1}$ is compact, we may assume after passing to a sub sequence and choosing appropriated representatives $(Z^{(m)}, \mathcal{A})$ of the projectivized stability conditions that the sequence $(Z^{(m)}(S_1), \ldots, Z^{(m)}(S_n))$ converges, in fact to a point Z_0 where precisely the first r_0 entries are different from zero by definition of $\succcurlyeq$. Since $Z^{(m)}(S_i) \in \overline{\mathbb{H}}$ we know that $Z_0(S_i) \in \overline{\mathbb{H}} \cup \mathbb{R}_{>0} \cup \{0\}$. Suppose that $Z_0(S_i) \in \mathbb{R}_{>0}$ for none of the S_i. Then we iterate the construction: We consider $(Z^{(m)}(S_{r_0+1}), \ldots, Z^{(m)}(S_n)) \in \mathbb{P}^{n-r_0-1}$ and use rescaling by real numbers and passage to a subsequence so that this converges to a point Z_1. If again $Z_1(S_i) \in \mathbb{R}_{>0}$ holds for none of the $S_i \in \mathrm{Sim}(\mathcal{A}_i)$ we continue to construct $Z_2, \ldots Z_L$, which we consider as functions $Z_i : K(\mathcal{A}_i) \to \mathbb{C}$. It is now obvious that the tuple $\sigma := (\mathcal{A}_\bullet, Z_\bullet)$ is a multi-scale stability condition and that

$\sigma_n \to \sigma$ by definition of the plumbing procedure (in fact plumbing with $\tau_i \in -i\mathbb{R}$ purely imaginary suffices as the lower levels just need to be rescaled appropriately).

Finally we have to deal with the case that $Z_i(S) \in \mathbb{R}_{>0}$ for some S we excluded so far. We'd like to rotate by some $\lambda \in S^1$ and apply Lemma 5.8. We have to be careful since this rotation has to be applied to $(Z^{(m)}, \mathcal{A})$ (not just the central charge) and might change the heart and alter our basic assumption. To circumvent this problem we use that for $\mathcal{D} = \mathcal{D}^3_{A_n}$ the heart $\mathcal{A}$ has only finitely many stables. Passing to a subsequence we may assume that there are only finitely many problematic phases $P \in S^1$ that arise as limits of phases ϕ such $\mathcal{P}_m(\phi) \neq \emptyset$ for the slicing $\mathcal{P}_m$ associated with σ_m and that (possibly iteratively) tilting at the torsion pairs defined by those slices we stay in the same fundamental domain of $\mathbb{P}\mathrm{MStab}(\mathcal{D}^3_{A_n})$ with respect to the action of G. Now we just apply the $\mathbb{C}$-action to the initial sequence for some λ such that $1 \notin e^{i\lambda}P$ and run the argument of the previous paragraph. □

Theorem 5.10, Propositions 5.13 and 5.14 together with Theorem 5.15 complete the *proof of Theorem 1.1.*

6. The BCGGM-compactification and examples

In this section we recall the main features and notions of the smooth compactification (as orbifold or DM-stack) of the strata of abelian differentials and quadratic differentials by multi-scale differentials, as constructed in [BCGGM3] together with [CMZ19] (see also [CGHMS23] for the log geometry viewpoint on this compactification). We focus on the case of the stratum Q_n corresponding to the A_n-quiver and denote the multi-scale compactification by $\overline{Q}_n$ and its projectivization by $\mathbb{P}\overline{Q}_n$. At the end of this section we will assert an isomorphism

$$\overline{K}_n : \mathbb{C}\backslash\overline{Q}_n/S_{n+1} \xrightarrow{\cong} \mathbb{C}\backslash \mathrm{MStab}^\circ(\mathcal{D}^3_{A_n})/\mathcal{A}ut^\circ(\mathcal{D}^3_{A_n}) \tag{40}$$

of complex orbifolds and give a sketch of proof.

As complex variety the labeled version of the projectivized stratum $\mathbb{P}Q_n$ is isomorphic to the moduli space $\mathcal{M}_{0,n+2}$ of pointed genus zero surfaces and as such comes with its Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n+2}$. This kind of compactification of quadratic differential strata is available only for genus zero differentials. One of the goals in this section is to explain why $\mathbb{P}\overline{Q}_n$ is *not isomorphic* to $\overline{\mathcal{M}}_{0,n+2}$.

Spaces of quadratic differentials. Let $\mathbf{w} = (w_1, \ldots, w_r)$ be a tuple of integers ≥ -1 and let $\mathbf{w}^- = (w_{r+1}, \ldots, w_{r+b})$ be a tuple of integers ≤ -2 in the quadratic case. Let $\mathrm{Quad}_{g,r+b}(\mathbf{w}, \mathbf{w}^-)$ be the moduli space of quadratic differentials $(X, \mathbf{z}, q)$ on a pointed curve $(X, \mathbf{z})$ where $\mathbf{z} = (z_1, \ldots, z_{r+b})$ such that q has signature $(\mathbf{w}, \mathbf{w}^-)$. In this space the critical points are labeled. The unlabeled version is denoted by $\mathrm{Quad}_g(\mathbf{w}, \mathbf{w}^-)$, i.e., without the subscript. We abbreviate

$$Q_n = \mathrm{Quad}_{0,n+2}(1^{n+1}, -n-3) \quad \text{and} \quad Q_{[n]} = \mathrm{Quad}_0(1^{n+1}, -n-3) = Q_n/S_{n+1}\,.$$

All these spaces come with their projectivized versions, the quotient by the $\mathbb{C}^*$-action, denoted by a letter $\mathbb{P}$ in front. Occasionally we will compare with spaces of abelian differentials, denoted by $\Omega\mathcal{M}_g(\mathbf{w}, \mathbf{w}^-)$. For a fixed weighted DMS $\mathbf{S_w}$ as in Section 3.2 we denote by $\mathrm{FQuad}(\mathbf{S_w})$ the *moduli space of framed quadratic differentials* $(X, \mathbf{z}, q, \psi)$ of signature $(\mathbf{w}, \mathbf{w}^-)$ with a (Teichmüller) marking ψ of the real oriented blowup of X at the poles by the surface $\mathbf{S_w}$. (We suppressed $\mathbf{w}^-$ in the notation.)

6.1. **Enhanced level graphs.** Recall (e.g. from [ACG11]) that boundary strata of the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ are indexed by the *dual graphs* of the corresponding stable curves.

Abelian case. The first datum to characterize points in a boundary stratum of the multi-scale compactification of $\Omega\mathcal{M}_g(\mathbf{w}, \mathbf{w}^-)$ is the following. An *enhanced level graph* $\widehat{\Gamma}$ *(for abelian differentials)* is the dual graph of a pointed stable curve $(\widehat{X}, \widehat{\mathbf{z}})$, with a weak total order on the vertices and a natural number κ_e, the *enhancement*, assigned with each edge. In particular $\widehat{\Gamma}$ is connected, unless specified otherwise. The weak total order is usually given arranging the vertices in levels, indexed by non-positive integers, the top level being level zero. We call an edge *horizontal* if it starts and ends at the same level, and *vertical* otherwise. We write $E = E(\widehat{\Gamma}) = E^h \cup E^v$ for this decomposition of the set of edges. We require that $\widehat{\kappa}_e = 0$ if and only if e is horizontal.

The enhancement encodes the orders of zeros and poles of the collection of differentials ω_v on the pointed stable curve $(\widehat{X}, \widehat{\mathbf{z}})$. On the vertex $v \in \widehat{\Gamma}$ the differential ω_v is required to have order w_i, if the i-th marked point $(i = 1, \ldots, r+b)$ is adjacent to v. At (the node corresponding to) a horizontal edge e adjacent to v the differential has a simple pole and the residues at the two ends of e match, i.e., they add up to zero. At the upper end of a vertical edge e the differential has a zero of order $\widehat{\kappa}_e - 1$, at the lower end a pole of order $-\widehat{\kappa}_e - 1$. In particular at each edge the orders add up to -2. Such a collection of differentials $\boldsymbol{\omega} = (\omega_v)$ is called a *twisted differential (of signature* $(\mathbf{w}, \mathbf{w}^-)$*) compatible with* $\widehat{\Gamma}$ if moreover the global residue condition (GRC) from [BCGGM1] holds. An enhanced level graph comes with a *vertex genus* g_v for each $v \in V$, defined by the requirement that $2g_v - 2$ is the sum of the adjacent zero and pole orders. For each signature there is only a finite number of enhanced level graphs (in particular a finite number of enhancements) for which the space of twisted differentials on each vertex is non-empty.

Quadratic differentials. We can view the space of quadratic differentials inside the space of abelian differentials (via the canonical cover construction) as a subspace of surfaces with an involution, see e.g. [CMZ19]. Due to the involutions only some of the (abelian) enhanced level graphs appear, encoded as follows. An *enhanced level graph* Γ *(for quadratic differentials)* is the dual graph of a stable curve $(X, \mathbf{z})$ with a level structure as above and enhancements κ_e associated with the edges $e \in E(\Gamma)$ with the only difference that we now aim for *twisted quadratic differentials* $\mathbf{q} = (q_v)$ compatible with Γ, which comprises the vanishing according to the signature $(\mathbf{w}, \mathbf{w}^-)$ at the points of $\mathbf{z}$ and the following three conditions: at horizontal edges q_v should have a double pole with matching 2-residues, at vertical edges the order are $\kappa_e - 2$ at the upper end and $-\kappa_e - 2$ at the lower end, and the collection $\mathbf{q}$ satisfies the global residue condition. This condition (see [BCGGM2] depends on a *double cover of enhanced level graphs* $\widehat{\pi} : \widehat{\Gamma} \to \Gamma$, which is a graph morphism with the following conditions. Edges with even κ_e have two preimages with enhancement $\widehat{\kappa}_e = \kappa_e/2$. Edges with odd κ_e have one preimage with enhancement $\widehat{\kappa}_e = \kappa_e$. The preimage of a vertex with an adjacent leg (marked point or edge) that carries an odd label is a single vertex. The preimage of a vertex without such an adjacent leg consists of two vertices, if the vertex genus is zero, and one or two vertices otherwise. Here the vertex genus g_v is defined by the requirement that $2(2g - 2)$ equals the sum of the adjacent zero and pole orders. (All these conditions are necessary for

$\widehat{\Gamma}$ to be an enhanced level graph compatible with a twisted differential $\boldsymbol{\omega} = (\omega_v)$ on a cover, abusively also denoted by $\widehat{\pi} : \widehat{X} \to X$ which is on each vertex v the canonical cover corresponding to q_v and such that $\widehat{\pi}^* q_v = \omega_v^2$.) See [BCGGM2] for an example where the double cover is not uniquely determined by Γ. Again, for given ρ the number of enhanced level graphs that allow a compatible $\mathbf{q}$ is finite. Figure 5 shows the double covers of enhanced level graphs for the boundary divisors where two resp. three simple zeros have come together.

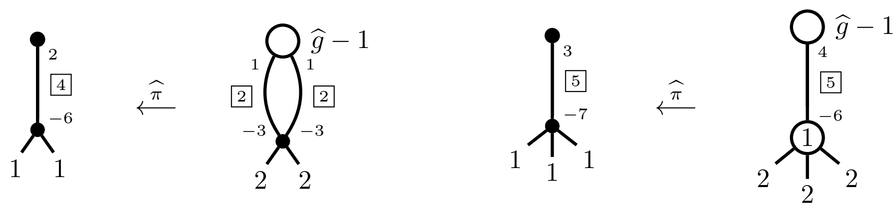


FIGURE 5. Level graphs of two (left) resp. three (right) zeros coming together and their double covers. Simple zeros and the pole of order $-n-5$ on top level are omitted. The boxed numbers are the κ_e.

Adjacency of boundary strata. For an enhanced level graph $\widehat{\Gamma}$ we denote by $D^{\circ}_{\widehat{\Gamma}}$ the open boundary stratum of multi-scale differentials (defined below) compatible with $\widehat{\Gamma}$. The boundary strata contained in the closure $D_{\widehat{\Gamma}}$ of $D^{\circ}_{\widehat{\Gamma}}$ can be described by the process of *degeneration*, or more easily starting with the converse process of *undegeneration*. Note that $D_{\widehat{\Gamma}}$ is in general not irreducible: the connected components of strata of meromorphic differentials that make up the twisted differential, or more generally components of strata with residue conditions, are one source that can create irreducible components

A *horizontal undegeneration* selects a subset H of the horizontal edges and contracts them. This results in a morphism δ^H of enhanced level graphs. To define the i-th *vertical undegeneration* view the i-th level passage as a line in the level graph just above level $-i$ and contract all the edges crossing that level passage. Again this results in a morphism δ_i of enhanced level graphs. This can obviously be generalized for any subset $I = \{i_1, \ldots, i_n\}$ of the set of levels to yield a graph contraction map δ_I. These two notions of undegeneration commute and a general undegeneration is a composition of the two. A degeneration of level graphs is the inverse procedure.

The complex codimension of a boundary stratum given by a level graph $\widehat{\Gamma}$ with L levels below zero and h horizontal edges is $h + L$.

Boundary strata of Q_n. For these type of strata the level graphs are strongly constrained.

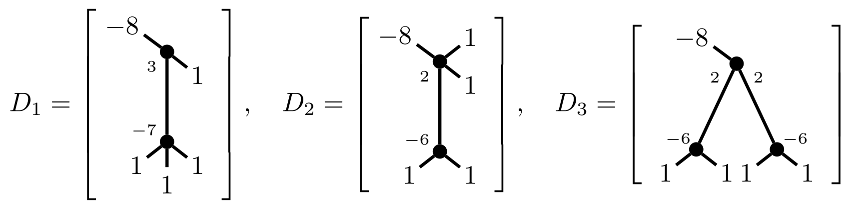
Lemma 6.1. *For the spaces Q_n i.e. with type $(\mathbf{w}, \mathbf{w}^-) = (1^{n+1}, -n-3)$ the level graphs are trees without horizontal edges and with all vertex genera $g_v = 0$. In particular for boundary strata of Q_n the graph Γ determines the double cover $\widehat{\Gamma}$.*

Proof. For the statement about horizontal edges, undegenerate all but one horizontal edge and all levels. Now note that each top level vertex must have at least one pole of order ≥ 2 or positive genus. For the second statement, the only ambiguity for $\widehat{\Gamma}$ given Γ is the 'criss-cross' ([BCGGM2, Example 4.3]), which requires $\pi_1(\Gamma) \neq \{e\}$. $\square$

6.2. **Examples.** We list the boundary components of $\mathbb{P}\overline{Q}_n$ and their adjacency in the two examples of lowest complexity.

The A_2-quiver. The projectivized space $\mathbb{P}\overline{Q}_2$ is a smooth compactification of $\mathcal{M}_{0,4}$. Since the construction introduces no orbifold structure in codimension one (see [BCGGM3, Section 6]), it agrees with $\overline{\mathcal{M}}_{0,4}$. The three boundary points correspond to the two-level graph with one edge, two vertices, and the pole together with one of the three simple zeros on top level. The action of S_3 permutes the three boundary points.

The A_3-quiver. In this case the projectivized labeled space $\mathbb{P}\overline{Q}_3$ is a surface, the largest dimension that can be visualized on a piece of paper. There are three types of boundary divisors, namely



The dual graphs are associated with labeled stable curves and this requires labeling the simple zeros. We distinguish the enhanced graph further by remembering the one (case D_1) or two (case D_2) simple zeros on top level, or the grouping in pairs at the end of each cherry (case D_3), see also Figure 6 where each of these boundary divisors occurs.

The codimension two strata are thus given by 'slanted cherries' (one of the lower ends of the level graph D_3 pushed down further to level -2), which are the intersection point of a D_3-divisor and a D_2-divisor, and a chain over three levels, with the pole and one zero on top, one on middle and two on bottom level, giving an intersection point of D_1 and D_2. It is easy to check that the boundary strata are all irreducible here, as depicted in Figure 6.

The action of S_4 is by permutation of the marked zeros and thus on boundary strata by the natural permutation action on the additional indices of each boundary divisor type D_i.

The Deligne-Mumford compactification. For comparison recall that boundary divisors of $\overline{\mathcal{M}}_{0,5}$ are in bijection with 2-element subsets of $\{1, \ldots, 5\}$. This shows that for the A_3-quiver the boundary divisors of $\overline{\mathcal{M}}_{0,n+2}$ are in bijection with those of type D_1 and D_2. There is a natural forgetful map $\mathbb{P}\overline{Q}_3 \to \overline{\mathcal{M}}_{0,5}$ that contracts the divisors of type D_3. The existence of 'cherry shaped' divisors like D_3 shows that $\mathbb{P}\overline{Q}_n$ is not isomorphic to $\overline{\mathcal{M}}_{0,n+2}$ for any $n \geq 3$. See [CGHMS23, Section 7] for more on this birational map.

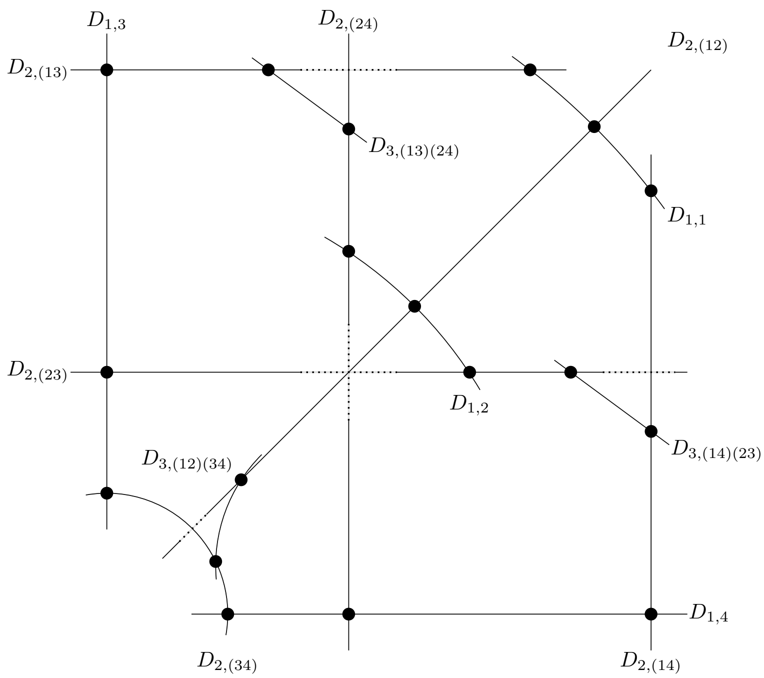


FIGURE 6. The boundary of the stratum $\mathbb{P}\overline{Q}_3 = \mathbb{P}\mathrm{Quad}_{0,5}(1^4, -8)$

6.3. **Multi-scale differentials.** We now give the key definition and explain the remaining terminology subsequently. A *quadratic multi-scale differential of type* $(\mathbf{w}, \mathbf{w}^-)$ on a stable pointed curve $(X, \mathbf{z})$ consists of

(i) an enhanced level structure on the dual graph Γ of $(X, \mathbf{z})$,
(ii) a twisted quadratic differential $\mathbf{q} = (q_v)_{v \in V(\Gamma)}$ of type $(\mathbf{w}, \mathbf{w}^-)$ compatible with the enhanced level structure,
(iii) and a prong-matching $\wp$ for each node of X joining components of non-equal level.

Two quadratic multi-scale differential are considered equivalent if they differ by the action of the level rotation torus.

We make the same definition for *abelian multi-scale differentials*, skipping the word 'quadratic' everywhere, replacing $\mathbf{q}$ by $\boldsymbol{\omega} = (\omega_v)$ and applying the abelian conventions for enhanced level graphs. To motivate the notion of 'prong-matching' and 'level rotation torus' we start with the

Proof of the isomorphism (40)*, Part I.* Our main goal is to define $\overline{K}_n^{-1}$ as a map of sets, starting with a multi-scale stability condition. If $\sigma = [\mathcal{A}_\bullet, Z_\bullet]$ is an honest stability condition, we associate with it a quadratic differential using the Bridgeland-Smith isomorphism recalled in Theorem 3.1.

Suppose from now on that σ is a strict multi-scale stability condition and suppose that the number of levels below zero is $L = 1$, leaving the bookkeeping for larger L to the reader. By Lemma 4.4 we may associate with $\mathcal{V} = \mathcal{V}_1^Z$ a type $\rho = (n_1, \ldots, n_{|J|})$ where J is an index set for the components of $\mathcal{V}$. We associate with σ the level graph Γ consisting of a tree with one vertex on top level (carrying the unique pole) and $|J|$ vertices on bottom level, each of them carrying $n_j + 1$ markings for simple zeros and enhancement $\kappa_j = n_j + 3$. As part of the bijectivity claim for the map $\sigma \mapsto (X, \mathbf{z}, \Gamma, \mathbf{q}, \wp)$ we are about to construct, we observe that all possible level graphs with $L = 1$ for quadratic differentials of type A_n arise in this way. We now apply the isomorphism from Theorem 3.4 (for the quotient category $\mathcal{D}/\mathcal{V}$) to the stability condition $(\overline{\mathcal{A}}_0, \overline{Z_0})$ on top level. We get the complex structure of the irreducible component of $(X, \mathbf{z})$ corresponding to the top level vertex v_0 together with the quadratic differential q_{v_0} on this components. Similarly we apply this isomorphism (for each $\mathcal{V}^j$) to each stability condition $(\mathcal{A}_1 \cap \mathcal{V}^j, Z_1|_{\mathcal{V}^j})$ on lower level to get the complex structure and the quadratic differential corresponding to the vertices v_j of Γ on lower level. (For $L = 1$ there is no further quotient, so we can as well apply Theorem 3.1 on lower level.) The enhancements of Γ were chosen so that the collection $\mathbf{q} = (q_v)_{v \in V(\Gamma)}$ is indeed a twisted differential compatible with the enhanced level structure. We let $\mathbf{z}$ be the unordered tuple of zeros and poles (different from the nodes of X) of the various differentials q_v. (We have no canonical way to label the points $\mathbf{z}$, and this fits with our target being the S_{n+1}-quotient of $\overline{Q}_n$.)

We need to be more precise about automorphisms in the application of Theorem 3.4 (or Bridgeland-Smith) at each level. In fact, above we were using that this isomorphism is equivariant with respect to the action of the mapping class group $\mathrm{MCG}(\mathbf{S}_\Delta)$ on the domain and of the group $\mathcal{A}ut^\circ(\mathcal{D})$ on the range (see [BMQS22, Theorem 7.2]). So far we have given a well-defined map $\sigma \mapsto (X, \mathbf{z}, \Gamma, \mathbf{q})$ with $\mathbf{z}$ considered up to the S_{n+1}-action. $\square$

The missing notions will be motivated by making this map bijective thanks to the prong $\wp$ and well-defined on equivalence classes. First observe that the above assignment depended on the stability conditions up to $G := \mathcal{A}ut^\circ_{\mathrm{lift}}(\mathcal{D}/\mathcal{V}) \times \prod_{j \in J} \mathcal{A}ut(\mathcal{V}^j)$. However the group fixing the boundary stratum of σ is $A := \mathcal{A}ut(\mathcal{D}, \mathcal{V})$ and its natural map $\varphi : A \to G$ is not surjective. In fact each $\mathcal{A}ut(\mathcal{V}^j)$ has an exact sequence (18), and the braid groups B_{n_j+1} for each j as well as $\mathcal{A}ut^\circ_{\mathrm{lift}}(\mathcal{D}/\mathcal{V})$ are in the image of φ, but the product of cokernels (each isomorphic to $\mathbb{Z}/(n_j+3)\mathbb{Z}$) is not hit surjectively. (Apply (18) to $\mathcal{A}ut(\mathcal{D}, \mathcal{V})$ and use that the cokernel is generated by the shift to prove this.) As a conclusion the equivalence relation generated by autoequivalences on multi-scale stability conditions is coarser than what is expected for Theorem 3.4 to be a bijection. We thus need an additional datum. To motivate the following definition, recall that the shift acts (via the correspondence to framed quadratic differentials) by cyclically shifting the marked points at each pole.

Prong matchings in the abelian case. A *prong* at a zero of order m of an abelian differential is a tangent vector that coincides with one of its $\kappa = m + 1$ outgoing horizontal directions. A *prong* at a pole of order $|m|$ is a tangent vector that coincides with one of its $\kappa = |m| - 1$ incoming horizontal direction. (For poles this is the same as choosing one of the marked points in M_i as defined in Section 3.1.) The prongs are labelled cyclically (by embedding in the plane) in clockwise order in the case of zeros (resp. counterclockwise order in the case of poles). Given an enhanced level graph $\widehat{\Gamma}$ a *prong-matching* $\widehat{\wp} = (\widehat{\wp}_e)_{e \in E^v}$ is a bijection of the prongs at the upper and lower even of each edge that reverses the cyclic order. Consequently, there are $\widehat{K}_{\widehat{\Gamma}} = \prod_{e \in E^v} \widehat{\kappa}_e$ different prong matchings for $\widehat{\Gamma}$.

The level rotation torus. The lower level differential should be projectivised, since only in this way limits are well-defined (compare with the proof of Theorem 5.15) and since only in this way the $\sigma \mapsto (X, \mathbf{z}, \Gamma, \mathbf{q})$ will pass to the equivalence class of σ. However just rescaling the lower level by $\mathbb{C}^*$ is no longer well-defined, as this comprises rotation and changes the horizontal direction that the notion of prong relies on. There is a finite unramified cover of $\mathbb{C}^*$ (of course: still abstractly isomorphic to $\mathbb{C}^*$) that naturally acts by rotation on the differentials $\boldsymbol{\omega}$ and on $\wp$ simultaneously so that the preimages of $1 \in \mathbb{C}^*$ fix the differential and permute cyclically each $\wp_e$. This algebraic torus (for general σ isomorphic to $(\mathbb{C}^*)^L$) is called *level rotation torus*, see [BCGGM3, Section 6] for the full definition.

Prong matchings and level rotation torus in the quadratic case. Prongs and their matchings are defined as in the abelian case, noting that a zero order m of a quadratic differential has $\kappa_e = m+2$ outgoing horizontal directions (to be counted on a local square root!) and a pole of order $|m|$ has $\kappa_e = |m|-2$ incoming horizontal directions. There are $K_\Gamma = \prod_{e \in E^v} \kappa_e$ prong matchings.

To understand the action of the level rotation torus, the easiest way is to pass to the canonical cover and use that a prong-matching of the quadratic differential induces a prong-matching of an abelian differential. Now the equivalence relation given by the level rotation torus is just defined as in the abelian case, restricted to those abelian multi-scale differentials that actually arise as double covers. See [CMS23, Section 7] for full details.

Proof of the isomorphism (40)*, Part II.* Finally we show how to associate with σ a prong-matching $\wp$. We continue with the setting above, in particular $L = 1$. Consider the ray obtained by plumbing $(it) * \sigma \in \operatorname{Stab}^\circ(\mathcal{D}_{A_n})$ with a purely imaginary parameter, i.e., without rotation. The limit $t \to \infty$ of the Bridgeland-Smith preimages $K^{-1}(it * \sigma)$ is a multi-scale differential with underlying $(X, \mathbf{z}, \Gamma, \mathbf{q})$ as above, by definition of plumbing and of the topology on $\overline{Q}_n$. It thus comes with a prong-matching $\wp$ and we now set $\overline{K}_n^{-1}(\sigma) = (X, \mathbf{z}, \Gamma, \mathbf{q}, \wp)$. (We remark that this $\wp$ is the only choice if we want $\overline{K}_n^{-1}$ to be continuous. Formally, in the language of [BCGGM3, Section 7] this is the only prong-matching so that the comparison diffeomorphisms between the welding of the limiting stable curve and the nearby plumbed curves is almost turning-number preserving. Informally, when $|J| \geq 2$ the right choice of $\wp$ differs from a wrong choice of $\wp'$ by rotating say one prong for the subsurface $j = 1$ on lower level. With the wrong $\wp'$ the turning numbers near the subsurface $j = 1$ do not work out. By rotating the whole lower level using the $\mathbb{C}$-action, turning numbers can be fixed for $j = 1$, but then since $|J| \geq 2$ the turning numbers will not work out at some other subsurface of lower level.)

To see that $\overline{K}_n^{-1}$ is bijective note that both the initial failure (due to the cokernel of $\varphi : A \to G$) and the additional datum $\wp$ capture the possibility of rotating a lower level component independently of other components. We leave the details to the reader.

To show continuity (and well-definedness mod $\mathcal{A}ut^\circ(\mathcal{D}^3_{A_n})$) it is best to first lift the map $\overline{K}_n^{-1}$ to a map from $\operatorname{MStab}^\circ(\mathcal{D}^3_{A_n})$ to the Teichmüller-framed version of $\overline{Q}_n$, the *augmented Teichmüller space* in the sense of [BCGGM3, Section 7]. This is a bordification of $\operatorname{FQuad}(\mathbf{S}_\mathbf{w})$ on which the mapping class group acts. One now needs to check that this lifted $\overline{K}_n^{-1}$ is a homeomorphism using the respective definition of topologies and the equivariance with respect to the mapping class group and $\mathcal{A}ut^\circ(\mathcal{D}^3_{A_n})$-action.

To show compatibility with the complex structure one needs to recall that the complex structure on $\overline{Q}_n$ is defined using plumbing (in the sense of complex geometry). This gives a collection of periods that defines local coordinates (the perturbed period coordinates in [BCGGM3, Section 9], in fact no modification of the differential is needed for A_n-type since all the residues are zero) and one only needs to check that they correspond to the coordinates defined in Section 5.3. We leave again the details to the reader. $\square$

6.4. **Why taking $\mathcal{A}ut^\circ(\mathcal{D}^3_{A_n})$-quotients? The A_2-quiver revisited.** In this subsection we revisit $\mathbb{P}\mathrm{MStab}(\mathcal{D}^3_{A_2})$ to show that prior to taking the $\mathcal{A}ut^\circ(\mathcal{D}^3_{A_n})$-quotient is neither a compact space (contrary to the Thurston-type compactifications in [BDL20]) nor carries a complex structure.

In fact $\mathbb{P}\mathrm{MStab}(\mathcal{D}^3_{A_2})$ coincides with the upper half plane with cusps $\widetilde{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1_{\mathbb{Q}}$ provided with the horoball topology where a neighborhood basis of ∞ consists of the sets $U_C = \{\tau : \mathrm{Im}(\tau) > C\}$ and a neighborhood basis of $z \in \mathbb{Q}$ are the images of U_C under a Möbius transformation mapping ∞ to z. Here $\mathbb{P}^1_{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ and it is known that $\mathbb{P}\,\mathrm{Stab}(\mathcal{D}^3_{A_2}) \simeq \mathbb{H}$, see e.g., [Sut11].

To prove this we start with a classification of the boundary strata. In this case necessarily $L = 1$. Since the Grothendieck group of a vanishing subcategory $\mathcal{V}$ of $\mathcal{D} := \mathcal{D}^3_{A_2}$ has rank 1, all stability conditions on it are projectively equivalent.

Next we list the possible $\mathcal{V}$. Recall that a heart suporting a stability condition on the space $\mathrm{Stab}(\mathcal{D})/\,\mathrm{sph}(\mathcal{D})$ can be identified with one of the following: the standard heart $\mathcal{H}_0 = \langle S_1, S_2\rangle$ or its shift $\mathcal{H}_0[1]$, or $\langle S_1[1], E\rangle$, $\langle S_1, S_2[1]\rangle$, $\langle S_2, E[1]\rangle$, where $S_2 \to E \to S_1$ is a short exact sequence. In fact, the vanishing category $\mathcal{A}_1$ arising from one of the hearts above is generated by one of the indecomposables S_1, S_2, E of $\mathcal{H}_0$, those appearing in Figure 1. Therefore in $\mathrm{Stab}(\mathcal{D})$, we associate with any such $\mathcal{V}$ the image of a generating simple in $\mathbb{P}^1(K(\mathcal{D})) \cong \mathbb{P}^1_{\mathbb{Q}}$ and call this map c. We let $\mathcal{H}_0$ be the standard heart of the A_2-quiver and let $\mathcal{V}_1 = \langle S_2\rangle$, corresponding to $c(\mathcal{V}_1) = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \in K(\mathcal{D})$. This subcategory obviously corresponds to any central charge with $Z_0(S_1) \in \pm\mathbb{H}$ and $Z_0(S_2) = 0$.

We first consider the action of the Seidel-Thomas group $\mathrm{sph}(\mathcal{D}) \cong B_3$ on $\mathcal{H}_0$ and $\mathcal{V}_1$. The element τ_2 stabilizes $\mathcal{V}_1$. The generator of center θ_2 of B_3 acts by the shift by $[\pm 5]$ and thus trivially on $\mathcal{V}_1$. Given that $B_3/\langle\theta_2\rangle \simeq \mathrm{PSL}_2(\mathbb{Z})$ [FM12] and that τ_2 acts as $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ on $K(\mathcal{D}^3_{A_2})$, the orbits of the action $\mathrm{sph}(\mathcal{D})$ on $\mathcal{V}_1$ are in bijection (of cosets) with

$$B_3/\langle\theta_2, \tau_2\rangle \cong \mathrm{PSL}_2(\mathbb{Z})/\langle\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\rangle \cong \mathbb{P}^1_{\mathbb{Q}} \tag{41}$$

The quotient $\mathrm{Aut}(\mathcal{D})/\,\mathrm{sph}(\mathcal{D})$ is generated by $[-1]$, which also acts trivially on any $\mathcal{V}$. To summarize, the orbit $\mathrm{Aut}(\mathcal{D}) \cdot \mathcal{V}_1$ is in natural bijection with $\mathbb{P}^1_{\mathbb{Q}}$ via the map c. The resulting space $\mathbb{P}\mathrm{MStab}(\mathcal{D})/\mathrm{Aut}(\mathcal{D})$ is the compact orbifold $\widetilde{\mathbb{H}}/\,\mathrm{PSL}_2(\mathbb{Z})$.

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