ON THE KODAIRA DIMENSION OF MODULI SPACES OF ABELIAN DIFFERENTIALS

DAWEI CHEN, MATTEO COSTANTINI, AND MARTIN MÖLLER

ABSTRACT. This paper lays the foundation for determining the Kodaira dimension of the projectivized strata of Abelian differentials with prescribed zero and pole orders in large genus. We work with the moduli space of multi-scale differentials constructed in [BCGGM2] which provides an orbifold compactification of these strata. We establish the projectivity of the moduli space of multi-scale differentials, describe the locus of canonical singularities, and compute a series of effective divisor classes. Moreover, we exhibit a perturbation of the canonical class which allows the corresponding pluri-canonical forms to extend over the locus of non-canonical singularities.

As applications, we certify general type for strata with few zeros as well as for strata with equidistributed zero orders when g is sufficiently large. In particular, we show general type for the odd spin components of the minimal strata for $g \geq 13$.

CONTENTS

1. INTRODUCTION

Let $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ be the moduli space of holomorphic (or meromorphic) Abelian differentials (up to scale) with labeled singularities of orders prescribed by a partition $\mu = (m_1, \ldots, m_n)$ of $2g-2$. This space is called the (projectivized) stratum of Abelian differentials of type μ.

Research of D.C. is supported by the National Science Foundation Grant DMS-2001040 and the Simons Foundation Collaboration Grant 635235.

Research of M.C. is supported by the DFG Research Training Group 2553.

Research of M.M. is supported by the DFG-project MO 1884/2-1, by the LOEWE-Schwerpunkt "Uniformisierte Strukturen in Arithmetik und Geometrie" and the Collaborative Research Centre TRR 326 "Geometry and Arithmetic of Uniformized Structures".

The study of $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ is important for at least two reasons. On one hand, an Abelian differential induces a flat metric with conical singularities at its zeros such that the underlying Riemann surface can be realized as a polygon with edges pairwise identified by translations. Varying the shape of such polygons by affine transformations induces an action on the strata of differentials (called Teichmüller dynamics), whose orbit closures (called affine invariant subvarieties) govern intrinsic properties of surface dynamics. On the other hand, an Abelian differential (up to scale) corresponds to a canonical divisor in the underlying complex curve. Hence the union of $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ stratifies the (projectivized) Hodge bundle over the moduli space of curves, thus producing a number of remarkable questions to investigate from the viewpoint of algebraic geometry, such as compactification, enumerative geometry, and cycle class calculation. The interplay of these aspects has brought the study of differentials to an exciting new era (see e.g., [Zor06], [Wri15], [Che17b] as well as the references therein for an introduction to this fascinating subject).

Despite the aforementioned advances, not much is known about the birational geometry of $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$. This is the focus of the current paper. A fundamental birational invariant for a variety is the Kodaira dimension, which measures the growth rate of pluri-canonical forms and controls the size of the canonical model of the variety. When the variety has a modular interpretation, determining the Kodaira dimension is closely related to the boundary behavior, singularity analysis, and decomposition of the cone of effective divisors. The study of the Kodaira dimension and related structures has covered many classical moduli spaces and their variants (see e.g., [HM82], [Tai82], [Har84], [CR86], [EH87], [O'G89], [CR91], [Kon93], [Li94], [Kon99], [Log03], [GHS07], [Far10], [FL10], [BFV12], [FV13a], [FV13b], [FV14], [CMKV17], [TVA19], [Pet19], [FJP20], [Sch20], [Sch21], [BM21], [AB21], [FJP21], [FM21], [FM22]). It is a general expectation that for sequences of moduli spaces the Kodaira dimension should be negative for small complexity (in terms of genus or level covering), but it should become maximal for large complexity. This is known, e.g., for moduli spaces of curves ([HM82]), for moduli spaces of abelian varieties ([Tai82]), and for moduli spaces of K3 surfaces ([GHS07]).

The (projectivized) strata of holomorphic Abelian differentials are uniruled for low genus $g \leq 9$ and all zero types μ as well as for $g \leq 11$ if moreover the number of zeros n is large ([Bar18], [Bud21]). On the other hand, when $n \geq g-1$ these strata can be viewed as generically finite covers of the moduli space of pointed curves ([Gen18]) and thus of general type for large genus.[1] However, in the case of few zeros the large genus behavior of the Kodaira dimension of the strata is wide open, which is one of the main motivations for the techniques developed in this paper.

In order to study the Kodaira dimension of a (non-compact) moduli space, one often needs a good compactification. The notion of multi-scale differentials from [BCGGM2] gives rise to two compactifications of the moduli stack $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$. First, the stack $\mathbb{P}\mathcal{MS}(\mu)$ of multi-scale differentials admits a local blowup description compared to the naive 'incidence variety compactification' ([BCGGM1]). Second, there is the smooth Deligne–Mumford stack $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ with normal crossing boundary divisors. They both have the same underlying coarse moduli space

[1] Strictly speaking in [Gen18] only the case $n = g-1$ was considered so as to obtain a generically finite map to $\mathcal{M}_g$. But the same argument works for $n > g-1$ by projecting to $\mathcal{M}_{g,n-g+1}$ and checking finiteness of the fiber over a boundary point parameterizing a general chain of elliptic curves with $n-g+1$ marked points in a tail.

$\mathbb{P}\mathrm{MS}(\mu)$. We will recall aspects of the relevant constructions and quotient maps in Section 2.

A standard method of showing general type is to write the canonical divisor class K as the sum of an ample divisor class and an effective divisor class (i.e., to prove that K is a big divisor class), where the existence of an ample divisor class already requires the underlying space to be projective. Note that $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ was constructed by a complex-analytic gluing approach, and that the blowup construction for the stack $\mathbb{P}\mathcal{MS}(\mu)$ is also local (i.e., a global ideal sheaf to be blown up is unknown in general). Such local operations might destroy the projectivity of the resulting complex-analytic varieties (see e.g., Hironaka's examples in [Har77, Appendix B]). Nevertheless, our first result below verifies the projectivity of $\mathbb{P}\mathrm{MS}(\mu)$.

Theorem 1.1. *The coarse moduli space $\mathbb{P}\mathrm{MS}(\mu)$ associated with the stack of multi-scale differentials of type μ is a projective variety for any μ.*

In order to prove the above result, in Section 3 we explicitly give a linear combination of boundary divisors that is relatively ample for the forgetful map from the multi-scale compactification to the incidence variety compactification (where the incidence variety compactification is projective since it is the closure of the strata in the projective Hodge bundle).

The canonical class of the stack $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ was computed in [CMZ20b, Theorem 1.1]. We then need to analyze the ramification divisor of the map from the stack to the coarse moduli space $\mathbb{P}\mathrm{MS}(\mu)$, both in the interior and at the boundary. This is carried out in Section 2.3. We remark that for strata of type $\mu = (m, 2g-2-m)$ with m even, the map to the coarse moduli space can actually have a ramification divisor in the interior.

We are now in a position to run the aforementioned strategy of expressing the canonical class as ample plus effective divisor classes. However, there are a number of new obstacles comparing to the work of Harris–Mumford for $\overline{\mathcal{M}}_g$ and subsequent works. The first one of them is about canonical singularities.

Theorem 1.2. *The interior $\mathbb{P}\Omega\mathrm{M}_{g,n}(\mu)$ of the coarse moduli space of Abelian differentials with labeled zeros and poles has canonical singularities for all signatures μ except for $\mu = (m, 2-m)$ in $g = 2$ with $1 \neq m \equiv 1 \bmod 3$.*

In contrast, the coarse moduli space of multi-scale differentials $\mathbb{P}\mathrm{MS}(\mu)$ has non-canonical singularities in the boundary for all but finitely many g.

A significant part of this paper deals with these non-canonical singularities and how to overcome their presence. By the Reid–Tai criterion, the absence of non-canonical singularities is certified by bounding from below the age of automorphisms acting on the tangent space of the moduli stack. At the boundary of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ the tangent space decomposes into two parts, as recalled in Section 4. One part is the tangent space of the strata determined by the vertices of the level graph at the corresponding boundary stratum. In Proposition 4.1 and Proposition 4.2 we compile tables listing the cases where the age is small enough to allow for non-canonical singularities. The other part in the tangent space describes the opening of nodes in terms of level passages (see Section 2.1 for the background on level graphs).

In order to control non-canonical singularities, as a preliminary step we recast the action of the stacky structure related groups on the level passages in terms of toric geometry. This was only implicitly described in [BCGGM2]. In Section 5 we explicitly determine the cone and fan structure to encode this information, which allows to measure the failure of a singularity from being canonical.

Next we define a non-canonical compensation divisor D_{NC} which is a linear combination of boundary divisors given explicitly in (28). It allows us to prove the following criterion (see Section 5.1 for relevant definitions and Section 7.5 for a refinement for strata with $\mu = (m, 2g-2-m)$ that takes the ramification divisor into account).

Proposition 1.3. *For $g \geq 2$ and all μ except for $\mu = (m, 2-m)$ in $g = 2$ with $1 \neq m \equiv 1 \bmod 3$, there exists an explicit effective divisor class D_{NC} such that pluri-canonical forms associated to the perturbed canonical class $K_{\mathbb{P}\mathrm{MS}(\mu)} - D_{\mathrm{NC}}$ in the smooth locus of $\mathbb{P}\mathrm{MS}(\mu)$ extends completely in a desingularization.*

In particular if one can write

$$K_{\mathbb{P}\mathrm{MS}(\mu)} - D_{\mathrm{NC}} \;=\; A + E \tag{1}$$

with A an ample divisor class and E an effective divisor class, then $\mathbb{P}\mathrm{MS}(\mu)$ is a variety of general type.

The description of D_{NC} builds on fairly delicate statements about automorphisms of small age acting on tangent spaces (see Proposition 4.11 and Proposition 5.11). Moreover, the description of D_{NC} reflects a strong tension: its divisor class has to be large enough to compensate non-canonical singularities, while it cannot be too large to make the desired expression (1) unrealizable (see Remark 5.14). Therefore, the ideas and techniques involved in the study of D_{NC} can be of independent interests for applications to other moduli spaces with non-canonical singularities.

We now turn to the class computation of effective divisors suitable to fulfill (1). The first type of divisors are pullbacks of a series of effective divisors from $\overline{\mathcal{M}}_{g,n}$ (such as divisors of Brill–Noether type). The formulas for such pullbacks and the conversion formulas between standard divisor classes λ_1, κ_1 and $\xi = c_1(\mathcal{O}(-1))$ on $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ are provided in Section 6. We remark that the verification of some pullback divisors not containing the entire stratum requires non-trivial degeneration techniques (e.g., using curves of non-compact type in the proof of Lemma 6.8). The second type of divisors, which are not induced via pullback, will be called *generalized Weierstrass divisors* and they can be defined for all connected components of the strata except hyperelliptic and even spin components. For a partition $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $g-1$ such that $0 \leq \alpha_i \leq m_i$ for all i we define the divisor in the interior as

$$W_\mu(\alpha) \;=\; \bigl\{(X, \mathbf{z}, \omega) \in \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu) \,:\, h^0\bigl(X, \alpha_1 z_1 + \cdots + \alpha_n z_n\bigr) \geq 2\bigr\} \tag{2}$$

in $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$. We will define the generalized Weierstrass divisor in $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ via a Porteous type setting and study its class, as well as its boundary behavior, in Section 7. Indeed we will use a novel twisted version of Porteous' formula in order to reduce extraneous contributions from the boundary divisors.

Using these divisor classes we can prove our main results on Kodaira dimensions for strata of various types. We start with the case when there is a unique zero.[2]

[2]A point z is called subcanonical if $(2g-2)z$ is a canonical divisor. Subcanonical points are among the most special points in algebraic curves. For $g \geq 4$ the locus of subcanonical points in $\mathcal{M}_{g,1}$ consists of three components, hyperelliptic, odd spin and even spin by [KZ03]. The hyperelliptic component of subcanonical points parameterizes Weierstrass points in hyperelliptic

Theorem 1.4. *The odd spin components of the coarse moduli spaces of Abelian differentials with a unique zero* $\mathbb{P}\Omega\mathrm{M}_{g,1}(2g-2)$ *are of general type for* $g \geq 13$.

Recall that the odd spin components of the minimal strata are known to be uniruled for $g \leq 9$. Hence in this case our result is nearly optimal. For the remaining cases of $g = 10$, 11 and 12, one needs either a more extremal effective divisor class or a certificate of non-general type. We remark that the proof of this theorem for $13 \leq g \leq 44$ relies on a computer verification of the constraints given by Proposition 1.3 (see the end of Section 8.6 for the description of the algorithm).

Next we treat the case of strata with 'few zeros'.

Theorem 1.5. *Given a constant* M*, consider all holomorphic signatures* $\mu = (m_1, \dots, m_n)$ *with even entries such that* $m_i \geq M$ *for all* i *and* $n \leq 5(M+1)$. *Then the odd spin components of the strata* $\mathbb{P}\Omega\mathrm{M}_{g,n}(\mu)$ *are of general type for all but finitely many such* μ.

In particular (for $M = 1$*), the odd spin components of the strata* $\mathbb{P}\Omega\mathrm{M}_{g,n}(\mu)$ *with at most* 10 *zeros are of general type for all but finitely many* μ.

Moreover for holomorphic signatures $\mu = (m_1, m_2)$ *with two odd entries, the (non-hyperelliptic) strata* $\mathbb{P}\Omega\mathrm{M}_{g,n}(\mu)$ *are of general type for all but finitely many* μ.

The above statement is aimed at simplicity. The precise condition on 'few zeros' for which we prove the theorem is given in Section 8.3. Note that the meaning of 'few zeros' is relative, e.g., an integer tuple close to $(\sqrt{g}^{\sqrt{g}}, g-2)$ with approximately $\sqrt{g}+1$ zeros is indeed a signature of 'few zeros'. On the other hand, the fact that for strata with zeros of odd order the range of our result is more limited is due to the constraint of the parameters α_i being integers in the definition of generalized Weierstrass divisors (since a natural choice for α_i is $m_i/2$).

Finally for strata with many zeros, our method can also be applied to the following zero type. We say that a stratum is equidistributed if the zero orders are all the same, i.e., $\mu = (s^n)$ with n entries of equal value s.

Theorem 1.6. *All but finitely many of connected equidistributed strata* $\mathbb{P}\Omega\mathrm{M}_{g,n}(s^n)$ *(and the odd spin components in the disconnected case) are of general type.*

The above result is again stated for simplicity rather than completeness. For instance, the result also holds for nearly equidistributed strata when μ is close to (s^n), e.g., for strata of type $(s-1, s+1, s^{n-2})$ with n large enough. All these results are proven by using divisors of Brill–Noether type and the generalized Weierstrass divisor for $\alpha = \mu/2$ (or a rounding of $\mu/2$ if there are zeros of odd order).

Contrary to the case of constructing effective divisors with low slope in $\overline{\mathcal{M}}_g$, a new phenomenon we have discovered is that it does not suffice to control the usual slope involving the boundary divisor Δ_{irr} (whose analogue in $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is the horizontal boundary divisor D_h). Instead, even for $\mu = (s^n)$ more boundary divisors are critical, e.g., boundary divisors consisting of 'vine curves' with two vertices and various numbers of edges. These boundary divisors impose tight bounds on the convex combination of divisors of Brill–Noether type and the generalized Weierstrass divisor for constructing the desired effective divisor class E in (1).

This paper opens the gate for exploring comprehensively the birational geometry of moduli spaces of differentials. In what follows we elaborate on further directions.

---

curves, hence it is a rational variety. In contrast, Theorem 1.4 reveals that the spin components of subcanonical points can behave very differently.

First, note that strata of holomorphic differentials with very unbalanced zero orders (such as $\mu = (g-2, 2^{g/2})$) are not covered by the current method of using a single generalized Weierstrass divisor (combined with a divisor of Brill–Noether type), which we will explain in Section 8. Nevertheless, we have obtained evidences that using a mixed version of generalized Weierstrass divisors might work (by varying the parameters α and taking a weighted average of all generalized Weierstrass divisors). The remaining challenges are the choice of weights, rounding α to be an integer tuple, and the combinatorial complexity of estimating.

Next, we have excluded the even spin components of the strata. This is due to the construction of the generalized Weierstrass divisor, e.g., for the minimal strata when $(2g-2)z$ is a canonical divisor of even spin, i.e., $h^0(X, (g-1)z) = 2$ (or a higher even number), the locus of $h^0(X, (g-1)z) > 1$ used for defining the generalized Weierstrass divisor would contain entirely the even spin component. A revised approach is to quotient out the (generically) two-dimensional subspace $H^0(X, (g-1)z)$ in the setting of Porteous' formula for the Hodge bundle. The remaining issue is caused by the locus where this subspace jumps dimension and hence the quotient can fail to be a vector bundle. Nevertheless, we expect that using certain blowup of this locus can help extend and complete the desired calculation.

Moreover, this paper deals exclusively with the strata where the zeros are marked. When there are zeros of the same order, an unmarked stratum is a finite quotient of the corresponding marked stratum induced by permuting the marked points of the same order, which can thus have distinct birational geometry. For instance, the unmarked principal stratum with $2g-2$ simple zeros is uniruled for all g because it is an open dense subset of the (projectivized) Hodge bundle, while the marked principal strata are of general type for large g. Many ideas and techniques in this paper can readily be adapted to treat the unmarked strata, e.g., in the singularity analysis in Proposition 4.1 leading to Theorem 5.1 we also consider the case of unmarked zeros and poles (since those with the same order can appear as indistinguishable edges in a level graph).

Another generalization is for strata of meromorphic differentials. This paper paves the way to treat them as well, e.g., the ramification divisor in Section 2, the projectivity of $\mathbb{P}\mathrm{MS}(\mu)$ in Section 3, and the singularity analysis in Section 4 and Section 5 cover the meromorphic case, too. It is interesting to note that the behavior of general type for meromorphic strata starts as early as for $g = 1$ from the corresponding geometry of modular curves.

Finally, our calculation for the classes of pullback divisors and generalized Weierstrass divisors provides the first step towards understanding the effective cone of $\mathbb{P}\mathrm{MS}(\mu)$, which together with the ample divisor class we constructed can shed light on the chamber decomposition of the effective cone and other birational models of $\mathbb{P}\mathrm{MS}(\mu)$. We plan to treat these questions in future work.

Acknowledgments. We thank Jonathan Zachhuber for his support in using the `diffstrata`-package that led to the proof of projectivity. We also thank Ignacio Barros, Andrei Bud, and Gavril Farkas for helpful discussions.

2. From the stack to the coarse moduli space

In this section we first recall some background on the geometry of the moduli stack of multi-scale differentials and its coarse moduli space. Our main goal here is to determine in Proposition 2.2 and in Proposition 2.5 the ramification loci of the map from the stack of multi-scale differentials to the coarse moduli space. Throughout we use $\mathrm{CH}^\bullet(\cdot)$ to denote the Chow groups with *rational* coefficients.

2.1. **The smooth Deligne–Mumford stack $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ and its boundary structure.** We recall some notation from [BCGGM2] and [CMZ20b], summarizing notions of multi-scale differentials and enhanced level graphs. For simplicity we often abbreviate $B = \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ and $\overline{B} = \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. We will also denote by $\varphi\colon \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) \to \mathbb{P}\mathrm{MS}(\mu)$ the map from the smooth Deligne–Mumford stack to the coarse moduli space.

Boundary strata in $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ are encoded by *enhanced level graphs*, which by definition are dual graphs of stable curves together with additional data. They are provided with a level structure, i.e. a total order on the vertices with equality permitted. Edges between vertices on the same level are called *horizontal*, and they are called *vertical* otherwise. Usually the top level is labeled by zero, and the levels below are labeled by consecutive negative integers. We also refer to the edges starting above level $-i$ and ending at or below level $-i$ as the edges crossing the *i-th level passage*. An *enhancement* is an assignment of an integer $p_e \geq 0$ to each edge, with $p_e = 0$ if and only if the edge is horizontal.[3] The enhancement encodes the number of prongs (real positive rays) emanating from the zeros and poles the multi-scale differentials have at the branches of the nodes corresponding to e. We usually omit 'enhanced' for level graphs.

Throughout the paper we rely on the nice boundary combinatorics of the moduli stack of multi-scale differentials. Many computations happen on boundary strata without horizontal nodes. We denote by $\mathrm{LG}_L(\overline{B})$ the set of enhanced level graphs with L levels below the top level and no horizontal nodes. We will also use the notation $\mathrm{LG}_L(\mu)$ in order to emphasize the signature, or simply LG_L when it is clear in the context. These level graphs correspond to subvarieties in the boundary of $\overline{B}$ with codimension L in $\overline{B}$. We denote by D_Γ the closed subvariety corresponding the boundary stratum with level graph Γ together with its degenerations. Each level of an enhanced level graph defines a generalized stratum ([CMZ20b, Section 4]). We denote by $d_\Gamma^{[i]}$ the projectivized dimension of level i. The normal crossing boundary structure implies that $\dim(\overline{B}) = L + \sum_{i=-L}^{0} d_\Gamma^{[i]}$ for every level graph Γ with L levels below zero.

Adjacency of boundary strata is encoded by the undegeneration map

$$\delta_{i_1,\ldots,i_n}\colon \mathrm{LG}_L(\overline{B}) \to \mathrm{LG}_n(\overline{B}), \tag{3}$$

which contracts all the passage levels of a non-horizontal level graph Γ except for the passages between levels $-i_k + 1$ and $-i_k$ for $k = 1, \ldots, n$. With this notation, $D_\Gamma \in \mathrm{LG}_L(\overline{B})$ is a union (due to prong-matchings) of connected components of the intersection of $\delta_j(D_\Gamma)$ for $j = 1, \ldots, L$. For $I = \{i_1, \ldots, i_n\}$ we also define $\delta_I^{\complement} = \delta_{I^\complement}$ for notation convenience.

Next we recall the notion of a *multi-scale differential*. This is a tuple $(X, \mathbf{z}, \boldsymbol{\omega}, \boldsymbol{\sigma}, \Gamma)$ consisting of a pointed stable curve $(X, \mathbf{z})$, a level graph Γ, a twisted differential $\boldsymbol{\omega}$ compatible with Γ and a collection of prong-matchings $\boldsymbol{\sigma}$. Here a twisted differential is a collection of differentials for each vertex of Γ with zeros and poles as

[3]These enhancements were denoted by κ_e in [BCGGM2]. We avoid that notation in view of the clash with constants derived from κ-classes. Our symbol reflects that these are the prongs of the differentials, comparing to [CMSZ20] where the same notation was used, but called 'twist'.

prescribed by the marked points and enhancements, subject to the residue conditions, as given more precisely in [BCGGM2]. The prong-matching $\boldsymbol{\sigma}$ is a collection of prong-matchings for each vertical edge, i.e. an orientation-reversing bijection of the prongs of the differential at the branches of the corresponding node. For simplicity we often omit certain part of the tuple $(X, \mathbf{z}, \boldsymbol{\omega}, \boldsymbol{\sigma}, \Gamma)$ when it is clear from the context.

The stack $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ parameterizes equivalence classes of multi-scale differentials, where two equivalent multi-scale differentials differ by the action of the *level rotation torus*. This is a (multiplicative) torus that acts simultaneously by rotating the differential and turning the prong-matching. The level rotation torus should be considered as the quotient of its universal covering $\mathbb{C}^L$ by the subgroup that fixes the differential on each level and brings all prongs back to themselves, where the subgroup is called the *twist group* and denoted by Tw_Γ. Not all elements in this group can be written as a product of twists that act on one level passage only, and those that can form the *simple twist group* Tw_Γ^s which is important since the normal crossing boundary structure of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ stems from compactifying each level passage. The quotient group $K_\Gamma = \mathrm{Tw}_\Gamma / \mathrm{Tw}_\Gamma^s$ is thus part of the stack structure as we see in the sequel. We call the elements of K_Γ the *ghost automorphisms.* We will frequently use that if $\Gamma \in \mathrm{LG}_1$ or if Γ has only horizontal nodes, then $K_\Gamma = \{e\}$ is trivial by definition.

2.2. **Coordinates at the boundary.** Recall from [BCGGM2] that a coordinate system near the boundary is given by perturbed period coordinates. Consider a boundary stratum D_Γ with L levels below zero, possibly also with horizontal nodes. Then the perturbed period coordinates around a multi-scale differential $(X, \boldsymbol{\omega}, \mathbf{z}, \boldsymbol{\sigma})$ compatible with Γ can be described as a product of three groups of coordinates:

- A parameter t_i parameterizing the opening-up of the level passage above level $-i$. We group them to a point $\mathbf{t} = (t_i) \in \mathbb{C}_{\mathrm{lev}} \cong \mathbb{C}^L$.
- The level-wise projectivized period coordinates $\mathbb{P}H^1_{\mathrm{rel}}(X_{(-i)})^{\mathfrak{R}_i}$ of the subsurfaces $X_{(-i)}$ on each level, where $\mathfrak{R}_i$ is the constraint imposed by the global residue condition to level $-i$. We define $\mathbb{A}_{\mathrm{rel}}(X_{(-i)}) \subseteq \mathbb{P}H^1_{\mathrm{rel}}(X_{(-i)})^{\mathfrak{R}_i}$ to be an affine chart containing the image of the level-wise flat surfaces $(X_{(-i)}, [\omega_{(-i)}])$ under the level-wise period coordinates, and denote by
$$\mathbb{A}_{\mathrm{rel}}(X) = \prod_{i=0}^{L} \mathbb{A}_{\mathrm{rel}}(X_{(-i)}) \tag{4}$$
the product of the level-wise affine charts.
- A parameter x_i for each horizontal node. We group them to a point $\mathbf{x} = (x_j) \in \mathbb{C}_{\mathrm{hor}} \cong \mathbb{C}^{H(\Gamma)}$ where $H(\Gamma)$ is the number of horizontal edges of Γ.

We revisit the map $\varphi\colon \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) \to \mathbb{P}\mathrm{MS}(\mu)$ near D_Γ. It can be factored first as a quotient by the group of ghost automorphisms K_Γ and then by the group $\mathrm{Aut}(X, \boldsymbol{\omega})$. An important conclusion from the construction of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is that the action of K_Γ on period coordinates is on $\mathbb{C}_{\mathrm{lev}}$ only and that $\mathrm{Aut}(X, \boldsymbol{\omega})$ maps prongs to prongs and thus acts on $\mathbb{C}_{\mathrm{lev}}/K_\Gamma$. As a result, a neighborhood U of $(X, \boldsymbol{\omega}, \mathbf{z}, \boldsymbol{\sigma})$ can be described as
$$(\mathbb{A}_{\mathrm{rel}}(X) \times \mathbb{C}_{\mathrm{hor}} \times \mathbb{C}_{\mathrm{lev}}/K_\Gamma)/\operatorname{Aut}(X, \boldsymbol{\omega}) \;\cong\; U \quad \subset \quad \mathbb{P}\mathrm{MS}(\mu)\,. \tag{5}$$

Remark 2.1. The exact sequence

$$0 \to K_\Gamma \to \mathrm{Iso}(X, \boldsymbol{\omega}) \to \mathrm{Aut}(X, \boldsymbol{\omega}) \to 0$$

describing the isotropy group in $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ of a multi-scale differential $(X, \boldsymbol{\omega})$ does not split in general. In particular, it is not a semidirect product in general. Consider for example a triangle graph (with three levels and one vertex on each level) with the prong $p_e = 2$ on the long edge e_2 and $p_e = 1$ on the short edges e_1 and e_3. Standard coordinates x_i and y_i that put the differentials at the upper and lower end of the edges in normal form are related to the level parameters t_i by ([BCGGM2, Equation 12.5])

$$x_1 y_1 = t_1^2, \qquad x_2 y_2 = t_1 t_2, \qquad x_3 y_3 = t_3^2 \,.$$

In this case $K_\Gamma \cong \mathbb{Z}/2$, comparing [CMZ20b, Example 3.3], as we will also retrieve in the sequel. Consider an automorphism of order two on the middle level, that acts by $y_1 \mapsto -y_1$ and $x_3 \mapsto -x_3$ while fixing the vertices on the other levels. This can be easily realized by a hyperelliptic involution, which moreover acts trivially on $\mathbb{A}_{\mathrm{rel}}(X)$ thanks to level-wise projectivization and $\mathbb{C}_{\mathrm{hor}}$ is void here. The lifts of this action to an action on $\mathbb{A}_{\mathrm{rel}}(X) \times \mathbb{C}_{\mathrm{hor}} \times \mathbb{C}_{\mathrm{lev}}$, i.e. to elements of $\mathrm{Iso}(X, \boldsymbol{\omega})$ are given by the action on $\mathbb{C}_{\mathrm{lev}}$, namely by

$$t_1 \mapsto \zeta_4^a t_1, \qquad t_2 \mapsto \zeta_4^{4-a} t_2, \qquad a \in \{1, 3\} \,.$$

The cases of $a = 1$ and $a = 3$ both have order four and differ by the action of the non-trivial element in K_Γ, thus ruling out the possibility of a splitting.

Here we analyze the action of $\mathrm{Aut}(X, \boldsymbol{\omega})$ on $\mathbb{C}_{\mathrm{lev}}/K_\Gamma$, in the simple case that Γ has two levels, which implies that K_Γ is trivial. We recall the essential step of the plumbing construction from [BCGGM2, Section 12]. At the upper and lower ends of each edge e of Γ we choose one pair (of the p_e possible choices) of coordinates x_e and y_e that puts the level-wise components $\omega_{(0)}$ and $\omega_{(-1)}$ of $\boldsymbol{\omega}$ in standard form and such that the collection of local prong-matchings $(dx_e \otimes dy_e)_{e \in E(\Gamma)}$ represents the global prong-matching $\boldsymbol{\sigma}$. Then the surfaces in a neighborhood are given by gluing in the plumbing fixture $x_e y_e = t^{m_e}$ where $m_e = \ell_\Gamma / p_e$ with $\ell_\Gamma = \mathrm{lcm}(p_e)_{e \in E(\Gamma)}$.

Suppose τ is an automorphism of $(X, \boldsymbol{\omega}, \mathbf{z}, \boldsymbol{\sigma})$, say mapping the edge e' to e (with $p_e = p_{e'}$). Then $\tau^* x_e$ is a coordinate near the upper end of e', which puts $\tau^* \omega_{(0)} = \zeta_{(0)} \omega_{(0)}$ in standard form. Consequently $\tau^* x_e / x_{e'} = \zeta_{e^+}$ for some root of unity ζ_{e^+} with $\zeta_{e^+}^{p_e} = \zeta_{(0)}$. Similarly, $\tau^* y_e / y_{e'} = \zeta_{e^-}$ for some root of unity ζ_{e^-} with $\zeta_{e^-}^{-p_e} = \zeta_{(-1)}$. The hypothesis that τ fixes the equivalence class of the prong-matching $\boldsymbol{\sigma}$ implies that there is some $c \in \mathbb{C}$ such that $\tau^* x_e \cdot \tau^* y_e = (ct)^{m_e}$ for all $e \in E(\Gamma)$. This c is in fact a root of unity and describes the action of τ on the coordinate t transverse to the boundary. If it exists, c is uniquely determined by

$$\zeta_{e^+} \cdot \zeta_{e^-} \;=\; c^{m_e} \quad \text{for all} \quad e \in E(\Gamma) \,. \tag{6}$$

2.3. **Ramification from the stack to the coarse moduli space.** In this section we are mainly interested in the map $\varphi \colon \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) \to \mathbb{P}\mathrm{MS}(\mu)$ from the smooth Deligne–Mumford stack to the coarse moduli space. We want to import intersection theory computations from [CMZ20b] on $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ and then pass to study the birational geometry of $\mathbb{P}\mathrm{MS}(\mu)$. We thus need to study the ramification divisor of this map.

We will also consider the factorization $\varphi = \varphi_2 \circ \varphi_1$ where $\varphi_1 \colon \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) \to \mathbb{P}\mathcal{MS}(\mu)$ is the map to the orderly blowup constructed in [BCGGM2] and where

$\varphi_2\colon \mathbb{P}\mathcal{MS}(\mu) \to \mathbb{P}\mathrm{MS}(\mu)$ is the map to its coarse moduli space. Since φ_1 is locally given by the map $[U/K_\Gamma] \to U/K_\Gamma$, where U is a neighborhood of a generic point in D_Γ and K_Γ was defined above as the group of ghost automorphisms, it implies that $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ and $\mathbb{P}\mathcal{MS}(\mu)$ have the same coarse moduli space and we thus get a factorization as claimed. Since K_Γ is trivial if $\Gamma \in \mathrm{LG}_1$ or if Γ has horizontal edges only, the map φ_1 has no ramification divisor.

Next we focus on φ_2. Recall that $\mathbb{P}\mathcal{MS}(\mu)$ is locally obtained as the normalization of the blowup of an ideal sheaf (the 'orderly blowup', see [BCGGM2, Section 7]) in the normalization of the incidence variety compactification, which by definition is the closure of strata in the projective Hodge bundle. Hence the isomorphism groupoids of $\mathbb{P}\mathcal{MS}(\mu)$ are contained in the isomorphism groupoids of $\overline{\mathcal{M}}_{g,n}$. They are given by the automorphism groups of pointed stable curves that respect the additional data encoded in the enhanced level graph, i.e., the enhancements need to be taken into account additionally.

From now on we will often encounter the notion of hyperelliptic differentials (X, ω) where X is hyperelliptic and ω is anti-invariant under the hyperelliptic involution. We remark that this notion is stronger than only requiring X to be a hyperelliptic curve. Moreover, a hyperelliptic component of a stratum means that the locus of hyperelliptic differentials forms a connected component of the stratum. Since in our setup the zeros and poles are labeled, among all hyperelliptic components only the ones for $\mu = (2g-2)$ (holomorphic) or $(2g-2+2m, -2m)$ with $m > 0$ (meromorphic) have the hyperelliptic involution as a non-trivial automorphism for a generic differential (up to sign) contained in them (in contrast for e.g. $\mu = (g-1, g-1)$ the hyperelliptic involution of a generic element in the hyperelliptic component swaps the two zeros). For this reason when analyzing the ramification of the map φ, we will exclude these special hyperelliptic components (whose birational geometry is much better known anyway, e.g. being unirational).

The result below describes the ramification divisor of the map $\varphi\colon \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) \to \mathbb{P}\mathrm{MS}(\mu)$ in the interior of the strata.

Proposition 2.2. *Suppose μ is of holomorphic type (and the hyperelliptic component is excluded if $\mu = (2g-2)$). Then the ramification divisor of the map φ in the interior of the stratum is empty unless $\mu = (m, 2g-2-m)$ consists of two zeros of even order (i.e. m is even). In this case the ramification divisor in the interior arises from the locus of canonical double covers of quadratic differentials in the stratum $\mathcal{Q}_{0,2g+2}(m-1, 2g-3-m, -1^{2g})$.*

Suppose μ is of (stable) meromorphic type (and the hyperelliptic component is excluded if $\mu = (2g-2+2m, -2m)$). Then the ramification divisor of the map φ in the interior of the stratum is empty unless $\mu = (m_1, m_2, 2g-2-m_1-m_2)$ consists of three zeros and poles of even order (i.e. m_1 and m_2 are both even). In this case the ramification divisor arises from the locus of canonical double covers of quadratic differentials in the stratum $\mathcal{Q}_{0,2g+2}(m_1-1, m_2-1, 2g-3-m_1-m_2, -1^{2g-1})$.

Proof. To determine the ramification divisor we only need to consider automorphisms stabilizing pointwise a divisorial locus in $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. We can moreover restrict to automorphism groups of prime order, since any non-trivial group has such a subgroup.

First consider $(X, \omega, z_1, \ldots, z_n) \in \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ in the stratum interior for $\mu = (m_1, \ldots, m_n)$. Let τ be an automorphism of X of prime order k, so that τ induces a cyclic cover $\pi\colon X \to Y$ of degree k with the quotient curve Y of genus h. Then

ω^k is τ-invariant, hence there exists a k-differential η in Y such that $\pi^*\eta = \omega^k$. The marked zeros and poles $z_1, \ldots, z_n$ of ω are fixed by τ, hence they are totally ramified under π. Suppose π has additional ramification points at $x_1, \ldots, x_\ell \in X$, each of which must also be totally ramified since k is prime. Then the signature of η is $(a_1, \ldots, a_n, 1-k, \ldots, 1-k)$ where $a_i = m_i + 1 - k$. We have the Riemann–Hurwitz relation

$$2g - 2 \;=\; k(2h-2) + (n+\ell)(k-1). \tag{7}$$

First suppose $\ell = 0$. For μ of holomorphic type (and hence $g \geq 1$), the projective dimension of the stratum of such (Y, ξ) is (at most) $2h-2+n$ (where the maximum dimension is attained if all $m_i + 1 - k \geq 0$). If $2h-2+n \geq 2g-3+n$, then $h \geq g$. But this is impossible for the branched cover π with at least one totally ramified point. This argument also works for μ of meromorphic type, using the inequality $2h-3+n \geq 2g-4+n$ instead.

Next suppose $\ell > 0$. Consider first the case that μ is of holomorphic type. Since $1-k < 0$, the projective dimension of the stratum of such (Y, η) is $2h-3+n+\ell$. Suppose $2h-3+n+\ell \geq 2g-3+n$, i.e., if such locus has at most codimension one in $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$. Then $2h + \ell \geq 2g$ and it follows from (7) that

$$(2k-2)h + (n-2)(k-1) + \ell(k-2) \leq 0\,.$$

- Suppose $k \geq 3$. Then $n = 1$, $h = 0$ and $\ell \leq 2$. Hence $2g \leq 2h + \ell \leq 2$ and $g \leq 1$. The only possibility is $g = 1$ and $\mu = (0)$, which gives the hyperelliptic component excluded in the assumption.
- Suppose $k = 2$. Then $2h + n - 2 \leq 0$, hence $h = 0$ and $n \leq 2$. If $n = 1$, then it gives the hyperelliptic component of the minimal stratum in genus g which is excluded in the assumption. If $n = 2$, then we obtain the locus of hyperelliptic differentials in the stratum $\mathbb{P}\Omega\mathcal{M}_{g,2}(m, 2g-2-m)$ that arises via canonical double covers from quadratic differentials in the stratum $\mathcal{Q}_{0,2g+2}(m-1, 2g-3-m, -1^{2g})$ of projectiveized dimension $2g-1$, i.e., of codimension one in the stratum $\mathbb{P}\Omega\mathcal{M}_{g,2}(m, 2g-2-m)$. In this case the two zeros are ramified, hence they are Weierstrass points. Consequently the zero orders m and $2g-2-m$ have to be even.

For μ of meromorphic type, the above inequalities become $2h + \ell \geq 2g - 1$ and $(2k-2)h + (n-2)(k-1) + \ell(k-2) \leq 1$. Moreover in this case $n \geq 2$, as a (stable) meromorphic differential has at least one zero and one pole. Then a similar analysis as above leads to the locus of hyperelliptic differentials in the meromorphic stratum $\mathbb{P}\Omega\mathcal{M}_{g,3}(m_1, m_2, 2g-2-m_1-m_2)$ with m_1 and m_2 both even.

Finally we have to make sure that the branching order of φ along the locus of hyperelliptic differentials in the stratum $\mathbb{P}\Omega\mathcal{M}_{g,2}(m, 2g-2-m)$ and in $\mathbb{P}\Omega\mathcal{M}_{g,3}(m_1, m_2, 2g-2-m_1-m_2)$ is just $k = 2$, not a higher power of two. This follows from the fact that a general hyperelliptic curve has no non-trivial automorphisms except the hyperelliptic involution. □

Remark 2.3. Using the definition of theta characteristics we can determine the spin parity of the differentials in the ramification divisors in Proposition 2.2. For a holomorphic differential of type $(2m, 2g-2-2m)$ with both zeros as Weierstrass points in the underlying hyperelliptic curve, the parity of the spin structure is given by $\lfloor m/2 \rfloor + \lfloor (g+1-m)/2 \rfloor \bmod 2$. For a meromorphic differential of type $(2m_1, 2m_2, -(2m_1+2m_2+2-2g))$ with both zeros and the pole as Weierstrass points in the underlying hyperelliptic curve, the parity is given by $\lfloor m_1/2 \rfloor + \lfloor m_2/2 \rfloor - \lfloor (m_1 + m_2 - g)/2 \rfloor \mod 2$.

To control the ramification at the boundary of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ we need a variant of the above proposition, allowing marked points to be permuted, but with automorphisms on the full stratum rather than just on a divisor.

Lemma 2.4. *Let μ be an arbitrary signature of a (stable) stratum of differentials, possibly of meromorphic type and with unlabeled singularities. If each ω in the stratum (component) $\mathbb{P}\Omega\mathcal{M}_{g,\{n\}}(\mu)$ admits a non-trivial automorphism τ of order k fixing ω up to a k-th root of unity, then $k = 2$, the automorphism τ is the hyperelliptic involution, and the stratum (component) is hyperelliptic for $\mu = (2g-2)$, $\{g-1, g-1\}$, $(2g-2+2m, -2m)$ with $m > 0$, $(2g-2+2m, \{-m, -m\})$ with $m > 0$ or $m < 1-g$, and $(\{m_1, m_1\}, \{-m_2, -m_2\})$ with $m_i > 0$ and $m_1 - m_2 = g - 1$.*

Proof. As before τ induces a cyclic cover $\pi\colon X \to Y$ of degree k with Y of genus h, and there exists a k-differential η in Y of signature μ' such that $\pi^*\eta = \omega^k$. By the Riemann–Hurwitz relation we have

$$2g - 2 \;=\; k(2h-2) + \sum_{i=1}^{b} (d_i - 1) r_i + \sum_{j=1}^{b'} (d'_j - 1) r'_j$$

where the singularities of ω are distributed into b orbits under τ, each having cardinality r_i with $d_i = k/r_i$, and in addition there are b' special (unmarked) orbits, each having cardinality $r'_j < k$ with $d'_j = k/r'_j$. (Note that we do *not* require $r_i < k$.) With these notations we have $n = \sum_{i=1}^{b} r_i$, hence the above relation can be rewritten as

$$2g - 2 + n \;=\; k(2h - 2 + b + b') - \sum_{j=1}^{b'} r'_j\,. \tag{8}$$

By assumption, the dimension of $\mathbb{P}\Omega\mathcal{M}_{g,\{n\}}(\mu)$ agrees with the dimension of the corresponding projectivized stratum of k-differentials $\mathbb{P}\Omega\mathcal{M}^k_{h,\{b+b'\}}(\mu')$, i.e. $2g - 2 + n = 2h - 3 + b + b'$ (if μ is of holomorphic type) or $2g - 3 + n = 2h - 3 + b + b'$ (if μ is of meromorphic type). In particular, it implies that $2h - 2 + b + b' \geq 2g - 2 + n$. Hence combining with (8) it gives

$$(k-1)(2h - 2 + b + b') \;\leq\; \sum_{j=1}^{b'} r'_j\,.$$

Since $k \geq 2$ and $r'_j \leq k/2$, we deduce that $(k-1)b' \geq kb'/2 \geq \sum_{j=1}^{b'} r'_j$. Then the preceding inequality implies that $(k-1)(2h - 2 + b) \leq 0$. Since $b \geq 1$, it follows that $h = 0$ and $b = 1$ or 2.

Suppose $b = 2$. Then $(k-1)b' \leq \sum_{j=1}^{b'} r'_j \leq kb'/2$. If $k > 2$, then $k - 1 > k/2$, and the only possibility is $b' = 0$. In this case $2g - 2 = -r_1 - r_2$, hence $g = 0$ and $r_1 = r_2 = 1$, which leads to the unstable signature $\mu = (-1, -1)$. If $k = 2$, then together with $h = 0$ and $b = 2$ we obtain those meromorphic hyperelliptic components as claimed.

Suppose $b = 1$. Then all $n = r_1$ singularities are in one fiber, i.e. $\mu = (m, \ldots, m)$. Moreover $(k-1)(b'-1) \leq \sum_{j=1}^{b'} r'_j \leq kb'/2$, i.e. $(k-2)(b'-2) \leq 2$. If $k > 2$, then

$b' \leq 4$. It follows that

$$2g - 2 + n \;=\; k(b'-1) - \sum_{j=1}^{b'} r'_j \;\leq\; b' - 1 \leq 3\,.$$

If $g = 0$, then $n = r_1 > 2$ by stability, but $mn = 2g-2 = -2$, leading to a divisibility contradiction. If $g = 1$, it is the case of elliptic curves with some ordinary markings, but the only non-trivial automorphism of a generic elliptic curve is the involution of order $k = 2$. The remaining case is $g = 2$ and $n = 1$, and consequently $k = 3$ and $b' = 4$, which occurs for $\mu = (2)$. But the unique zero z is a Weierstrass point and $3z$ cannot be a fiber of a cyclic triple cover of $\mathbb{P}^1$ since z is a base point of the line bundle $\mathcal{O}(3z) \cong K(z)$. Finally if $k = 2$, since $h = 0$ and $b = 1$, it leads to those holomorphic hyperelliptic components as claimed. $\square$

We can now describe the ramification at the boundary. For holomorphic signatures μ, the following boundary strata will be used for this purpose. We say that a two-level graph $\Gamma \in \mathrm{LG}_1(\mu)$ is a *hyperelliptic bottom tree (HBT)*, if it is a tree and all the vertices on bottom level belong to a stratum with signature $(2m_0, -2m_1, \ldots, -2m_\ell)$ for some integers $m_i > 0$. In particular, every bottom vertex has exactly one labeled zero. For an HBT graph Γ we denote by $D_\Gamma^{\mathrm{H}} \subset D_\Gamma$ the union of irreducible components where the differentials on each vertex on bottom level admit a hyperelliptic involution which fixes the labeled zeros and poles (i.e. the edges of the graph, and hence in particular the residues at the edges are all zero, thus satisfying the GRC as required in [BCGGM2]). In particular, these bottom differentials in D_Γ^{H} are hyperelliptic, i.e. they are anti-invariant under the hyperelliptic involution.

We say that a two-level graph $\Gamma \in \mathrm{LG}_1(\mu)$ is a *hyperelliptic top backbone (HTB)*, if it is a tree with a unique bottom vertex (i.e. a backbone graph) such that every top vertex is of type $(2g_i - 2)$ where g_i is the genus of the vertex. In particular, all labeled zeros are on bottom level. For an HTB graph Γ we denote by $D_\Gamma^{\mathrm{H}} \subset D_\Gamma$ the union of irreducible components where the differentials on each vertex on top level belong to the hyperelliptic component of the stratum with signature $(2g_i - 2)$.

Note that a graph $\Gamma \in \mathrm{LG}_1(\mu)$ for holomorphic signature μ can be of type HBT and HTB at the same time only if $\mu = (2g-2)$, and in that case Γ is a backbone graph with a unique bottom vertex carrying the unique labeled zero.

We will also encounter graphs $\Gamma \in \mathrm{LG}_1(2g-2)$ that we call *hyperelliptic banana backbones (HBB)*, where Γ has a unique bottom vertex (carrying the unique zero), there exists an involution τ fixing the vertices of Γ, the signature at each vertex admits a hyperelliptic component (taken the GRC into account for the bottom vertex) with τ as the hyperelliptic involution, and the quotient graph by τ is a backbone graph. In particular, the edges of Γ are either fixed or pairwise permuted by τ (where each permuted pair of edges looks like a banana in the drawing). We further require an HBB graph to contain at least one banana (otherwise it is of type HBT and HTB).

For an HBB graph Γ we denote by $D_\Gamma^{\mathrm{H}} \subset D_\Gamma$ the union of irreducible components where the differentials on each vertex belong to the hyperelliptic component and where moreover the prong-matchings are chosen so that the hyperelliptic involution does not extend to a neighborhood (see (6) and the surrounding paragraphs in Section 2.2 below for more details). We remark that without considering prongs the hyperelliptic component and the spin component of a reducible stratum can actually intersect along the boundary (see [Gen18, Corollary 7.10] and [Che17a, Theorem 5.3] for an example).

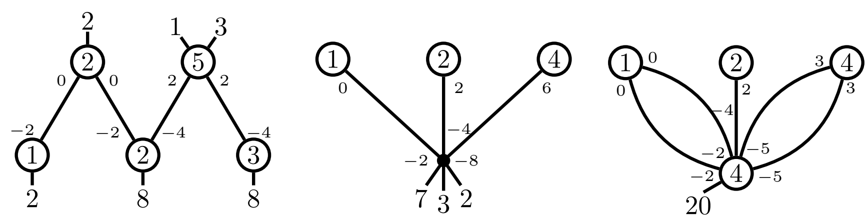


FIGURE 1. A hyperelliptic bottom tree graph (HBT), a hyperelliptic top backbone graph (HTB), and a hyperelliptic banana backbone graph (HBB).

Proposition 2.5. *Suppose μ is of holomorphic type (and the hyperelliptic component is excluded if $\mu = (2g-2)$). At boundary divisors the map $\varphi\colon \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) \to \mathbb{P}\mathrm{MS}(\mu)$ is ramified at most of order two.*

More precisely, φ is ramified at the components D_Γ^{H} for Γ of type HTB or HBT (except for $\mu = (2g-2)$ and Γ of both HTB and HBT types), φ is ramified at the components D_Γ^{H} for Γ of type HBB (where the hyperelliptic involution does not extend to a neighborhood), and φ is not ramified at any other components of the boundary.

In particular, φ is unramified at the horizontal boundary divisor D_h.

For meromorphic signatures μ, the ramification situation can be similarly described at the boundary. We add a prime, e.g. HTB', to denote the corresponding types of graphs in the meromorphic case, with some extra allowances or requirements as follows. HTB' graphs allow additional meromorphic signatures of type $\mu_i = (-2m_i, 2g_i - 2 + 2m_i)$ and $(-2m_i, \{g_i - 1 + m_i, g_i - 1 + m_i\})$ for top level vertices. HBB' graphs occur for meromorphic signatures $\mu = (-2m, 2g-2+2m)$ with the marked zero on the unique bottom vertex and the marked pole on one of the top vertices. HBT' graphs require all marked poles to concentrate on one top vertex (and the other top vertices are of holomorphic type).

Proposition 2.6. *Suppose μ is of meromorphic type (and the hyperelliptic component is excluded if $\mu = (-2m, 2g-2+2m)$). At boundary divisors the map $\varphi\colon \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) \to \mathbb{P}\mathrm{MS}(\mu)$ is ramified at most of order two.*

More precisely, φ is ramified at the components D_Γ^{H} for Γ of type HTB' or HBT' (except for $\mu = (-2m, 2g-2+2m)$ and Γ of both HTB' and HBT' types), φ is ramified at the components D_Γ^{H} for Γ of type HBB' (where the hyperelliptic involution does not extend to a neighborhood), and φ is not ramified at any other components of the boundary.

In particular, φ is unramified at any horizontal boundary divisor.

In the proof below we will use the following observation implicitly. Suppose ω is anti-invariant under the hyperelliptic involution τ, i.e. $\tau^*\omega = -\omega$. Then the residue of ω is zero at any hyperelliptic Weierstrass point, and the sum of the residues of

ω is zero at any pair of hyperelliptic conjugate points. In particular, if the GRC imposes the residue-zero condition to an edge of Γ fixed by τ or to a pair of edges swapped by τ (i.e. a banana), then this residue condition is automatically satisfied by ω.

Proof of Proposition 2.5 and Proposition 2.6. For holomorphic signatures μ, consider first the horizontal divisor D_h given by $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu,-1,-1)$ identifying the simple poles p_1 and p_2 to a node q. If an automorphism τ fixes the generic point in this boundary divisor up to a rescaling of ω, then Lemma 2.4 implies that $\mu = (2g-2)$, a case which has been excluded.

Second, for meromorphic signatures μ, there can be another type of horizontal divisors where the horizontal edge corresponds to a separating node. But in that case one component separated by the node must admit a non-trivial automorphism that fixes the simple pole at the node, which is impossible by Lemma 2.4.

Next we treat boundary divisors D_Γ for $\Gamma \in \mathrm{LG}_1(\mu)$. If D_Γ is in the ramification locus of φ, then every multi-scale differential $(X, \boldsymbol{\omega} = (\omega_{(0)}, \omega_{(-1)}), \boldsymbol{\sigma})$ in D_Γ admits a non-trivial automorphism τ. Since every top level vertex has either positive genus (with generically distinct moduli) or contains a labeled pole, τ cannot permute top level vertices. Similarly every bottom level vertex contains a labeled zero, hence τ cannot permute them either. Therefore, τ acts as an automorphism on each vertex of Γ.

Recall from [BCGGM2] that projectivized multi-scale differentials are represented by $(X, \boldsymbol{\omega}, \boldsymbol{\sigma})$ up to projectivization (rescaling all levels simultaneously). Moreover, the action of the level rotation torus rescales $\omega_{(-1)}$ and acts on the prong-matching $\boldsymbol{\sigma}$ simultaneously. Suppose that τ has order $k_{(i)}$ when restricted to level i. We conclude that $\tau^*\omega_{(i)} = \zeta_{(i)}\omega_{(i)}$ for $i = 0, -1$, where $\zeta_{(i)}$ is a $k_{(i)}$-th root of unity (not necessarily primitive). We remark that in the sequel a non-trivial action on a vertex v means τ restricted to the underlying marked surface X_v is non-trivial, and it does *not* necessarily imply that $\zeta \neq 1$ (but if $\zeta \neq 1$ then clearly τ must act non-trivially on X_v). We also make a useful observation that if τ does not fix every edge of Γ, then τ acts non-trivially on some vertices in both levels.

Consider first the case that τ acts non-trivially on the top level and we enumerate the vertices on that level by X_i. By Lemma 2.4, the action of τ restricted to each top level vertex, if non-trivial, can only be the hyperelliptic involution which maps ω to $-\omega$ on that vertex. Since the top level differentials are projectivized simultaneously, if τ induces a hyperelliptic involution on one top level vertex, then it must act in the same way for every top vertex. Therefore, if μ is a holomorphic signature, each top level vertex carries a hyperelliptic differential of type $\mu_i = (2g_i - 2)$ or $\{g_i - 1, g_i - 1\}$, where g_i is the genus of the vertex, i.e. it has no labeled zero and admits either one edge fixed by τ or a pair of edges (i.e. a banana) interchanged by τ. Similarly if μ is a meromorphic signature, we allow in addition meromorphic signatures of type $(-2m_i, 2g_i - 2 + 2m_i)$ and $(-2m_i, \{g-1+m_i, g-1+m_i\})$ for top level vertices. In both cases Γ is a banana tree with a unique bottom vertex that contains all labeled zeros. If τ acts trivially on the bottom level, then there is no banana, and we get HTB and HTB' (hyperelliptic top backbone) in the holomorphic and meromorphic cases, respectively. If τ acts non-trivially on the unique bottom vertex (e.g. if there exists at least one banana), one can verify (by a similar but simpler argument as in the next paragraph) that the bottom (generalized) stratum must be hyperelliptic with a unique labeled zero, and all labeled poles belong to one top level vertex. This case corresponds to HBB and HBB' (hyperelliptic banana backbone) with $\mu = (2g-2)$ in the holomorphic case and $\mu = (-2m, 2g-2+2m)$ in the meromorphic case, respectively.

Next suppose τ acts trivially on the top level and non-trivially on the bottom level, and we enumerate now the vertices on bottom level by X_i. In particular, all edges are fixed by τ. Suppose the bottom level has $v^\perp$ many vertices X_i, each of genus g_i, with n_i labeled zeros and poles, and e_i edges. We separate the discussion in two cases. Consider first the case $\tau^*\omega_i = \zeta_{(-1)}\omega_i$ for some $\zeta_{(-1)} \neq 1$ for all differentials ω_i on X_i in the bottom level. This assumption implies that τ restricted to the marked surface X_i is a non-trivial action of order $k_i \geq 2$. Let Y_i be the quotient of X_i by τ and denote its genus by h_i. Suppose that $\pi_i \colon X_i \to Y_i$ has in addition ℓ_i branch points (not from the images of labeled zeros and poles and edges), and that over each such branch point the fiber cardinality is $c_{i,j}$ with multiplicity $d_{i,j}$ for each fiber point, i.e. $c_{i,j}d_{i,j} = k_i$ with $c_{i,j} \leq k_i/2$. Then we have the Riemann–Hurwitz relation

$$2g_i - 2 + n_i + e_i \;=\; k_i(2h_i - 2 + n_i + e_i + \ell_i) - \sum_{j=1}^{\ell_i} c_{i,j}\,.$$

For a holomorphic signature μ, the bottom generalized stratum has (unprojectivized) dimension equal to

$$N^\perp \;=\; \Big(\sum_{i=1}^{v^\perp}(2g_i - 2 + n_i + e_i)\Big) - (v^\top - 1)\,,$$

where $v^\top$ is the number of top vertices, and we subtract $v^\top - 1$ because the GRC imposes this many independent conditions (besides the Residue Theorem condition on each vertex). For a meromorphic signature μ, the bottom generalized stratum has (unprojectivized) dimension bigger than or equal to the above formula, since a top vertex with marked poles does not impose a GRC, and the equality is attained if and only if all top level marked poles belong to the same vertex (so the GRC is imposed by the other $v^\top - 1$ holomorphic top vertices independently). The dimension of the (unprojectivized) locus of those Y_i is

$$N' \;=\; \sum_{i=1}^{v^\perp}(2h_i - 2 + n_i + e_i + \ell_i)\,.$$

By assumption we have $N^\perp \leq N'$, which implies that

$$\sum_{i=1}^{v^\perp}(k_i - 1)(2h_i - 2 + n_i + e_i + \ell_i) \leq v^\top - 1 + \sum_{i=1}^{v^\perp}\sum_{j=1}^{\ell_i} c_{i,j}\,.$$

Since $k_i \geq 2$ and $c_{i,j} \leq k_i/2$, we have

$$\sum_{i=1}^{v^\perp}\sum_{j=1}^{\ell_i} c_{i,j} \leq \sum_{i=1}^{v^\perp}(k_i/2)\ell_i \leq \sum_{i=1}^{v^\perp}(k_i - 1)\ell_i\,.$$

Moreover since Γ is connected, we have $\sum_{i=1}^{v^\perp} e_i \geq v^\top - 1 + v^\perp$, where the equality holds if and only if Γ is a tree graph. It follows that $\sum_{i=1}^{v^\perp}(2h_i - 1 + n_i) \leq 0$. Since every $n_i > 0$, the only possibility for $2h_i - 1 + n_i \leq 0$ is $(h_i, n_i) = (0, 1)$ for which $2h_i - 1 + n_i = 0$, and hence all inequalities involved above must be equalities. Therefore, we conclude that all $k_i = 2$, $h_i = 0$, $n_i = 1$, $c_{i,j} = 1$, and that Γ is a tree graph with each bottom vertex carrying a hyperelliptic differential and having exactly one labeled zero, and any two adjacent top and bottom vertices are joined by a single edge fixed under the hyperelliptic involution of the bottom vertex. We thus conclude that this case corresponds to HBT and HBT' (hyperelliptic bottom tree) in the holomorphic and meromorphic cases, respectively.

Now consider the other case when τ acts trivially on the top level, non-trivially on the bottom level, but $\tau^*\omega_i = \omega_i$ for all differentials ω_i on the lower level vertices. Using the above notation for the quotient map, ω_i being τ-invariant implies that $\omega_i = \pi_i^*\eta_i$ for an Abelian differential η_i in each Y_i. Moreover, since all edges and labeled zeros and poles are fixed by τ, the quotient differential (Y_i, η_i) has the same number of zeros and poles as (X_i, ω_i) (despite possibly a different genus and different orders of zeros and poles). We also infer that the residue of ω_i at any polar edge is equal to k_i times the residue of η_i at the image pole under π_i. For the purpose of dimension count we can thus replace each (X_i, ω_i) by $(Y_i, k_i\eta_i)$ in the bottom level of Γ, as residue constraints imposed by the GRC are linear and depend only on the graph topology of Γ, not its decoration by genera and enhancements. It follows that the GRC imposes the same number of conditions to the lower level of the graph before and after the replacement. Hence by assumption we conclude that

$$\sum_{i=1}^{v^\perp}(2g_i - 2 + n_i + e_i) \;=\; \sum_{i=1}^{v^\perp}(2h_i - 2 + n_i + e_i)\,.$$

By the Riemann–Hurwitz relation

$$2g_i - 2 + n_i + e_i = k_i(2h_i - 2 + n_i + e_i) \geq 2h_i - 2 + n_i + e_i$$

and the inequality is strictly if $k_i > 1$. But there exists at least one $k_i > 1$ by the assumption that τ acts non-trivially on the bottom level, thus leading to a contradiction to the preceding identity.

The ramification orders at the components D_Γ^{H} for those special graphs Γ will be justified in Section 2.2 below. $\square$

Proof of Proposition 2.5 and Proposition 2.6, ramification orders. For simplicity we use the graph notations for holomorphic signatures μ. The argument works identically for the case of meromorphic signatures. We start with the case of $(X, \boldsymbol{\omega}, \mathbf{z}, \boldsymbol{\sigma})$ in a component D_Γ^{H} for Γ of type HBT or HTB, i.e., the involution τ fixes all the edges. In particular $\zeta_{e^\pm}^2 = 1$ for every edge e. Moreover, p_e is odd for all e, hence $\zeta_{e^\pm} = -1$ if and only if the action on the corresponding level is non-trivial. This implies that the ramification order is two at D_Γ^{H} for these graphs, except for the simultaneous involution of the intersection of HBT and HTB, where the action on the t parameter is given by multiplication by $c = (-1)(-1) = 1$ (this is because the system of equations (6) has a unique solution and $c = 1$ is a valid one), and hence the map is not ramified at D_Γ^{H} when Γ is of both HBT and HTB types.

Next consider an HBB graph Γ. If an edge e is fixed by the involution τ, then p_e is odd and hence as above $\zeta_{e^+}\zeta_{e^-} = (-1)(-1) = 1$. Suppose two edges e_1 and e_2 are swapped. Being an involution implies that $\zeta_{e_1^+}\zeta_{e_2^+} = 1 = \zeta_{e_1^-}\zeta_{e_2^-}$. Since these two edges have the same p_e and hence the same m_e, the system (6) is solvable only if $\zeta_{e_1^+}\zeta_{e_1^-} = \zeta_{e_2^+}\zeta_{e_2^-}$. Since τ is an involution, $c^2 = 1$, and hence $\zeta_{e_i^+}\zeta_{e_i^-} = \pm 1$.

Moreover, any component of D_Γ is either unramified (if $c = 1$) or has ramification order two (if $c = -1$), depending on whether or not the hyperelliptic involution on the boundary component can extend to a neighborhood in the interior.

If p_{e_i} is even, then indeed both possibilities can occur. Suppose for simplicity that there are no other edges besides the swapped pair. Given a solution of the above system with $\zeta_{e_i^+}\zeta_{e_i^-} = -1$, we can replace the coordinate x_1 by $-x_1$. This still puts $\omega_{(0)}$ in standard form there (as $d(x_1^{p_{e_1}})$ is unchanged), but swaps the sign of both $\zeta_{e_i^+}$ and thus the new roots of unity satisfy that $\zeta_{e_i^+}\zeta_{e_i^-} = +1$.

On the other hand, if all the twists p_e are odd, then ℓ_Γ is odd. Since $\zeta_{(0)} = \zeta_{(-1)} = -1$, raising (6) to the p_e-th power implies that $c^{\ell_\Gamma} = (-1)(-1) = 1$ and hence $c \neq -1$. In this case the hyperelliptic involution always extends to the interior, i.e., the corresponding D_Γ lies in the boundary of the hyperelliptic component. □

3. Projectivity of the coarse moduli space

In this section we recall the background on the geometry of the moduli stack of multi-scale differentials and its coarse moduli space. The main goal is to prove the projectivity announced in Theorem 1.1.

Denote by $\overline{\mathcal{M}}_{g,n}(\mu)$ the closure of the (projectivized) stratum of Abelian differentials of type μ in the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$, and by $\overline{M}_{g,n}(\mu)$ its coarse moduli space. This is the image of a projection of the incidence variety compactification originally defined in [BCGGM1], where the projection contracts boundary strata whose level graphs have at least two vertices on top level. There is a forgetful morphism of stacks $f\colon \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) \to \overline{\mathcal{M}}_{g,n}(\mu)$, and we denote by $\overline{f}\colon \mathbb{P}\mathrm{MS}(\mu) \to \overline{M}_{g,n}(\mu)$ the corresponding map of coarse moduli spaces. The space $\overline{M}_{g,n}(\mu)$, as a subvariety of $\overline{M}_{g,n}$, is projective and thus has an ample line bundle $\mathcal{A}$. Recall that a line bundle $\mathcal{B}$ is called *$\overline{f}$-ample* or *relatively ample*, if $\mathcal{B}$ is ample on every fiber of $\overline{f}$. If $\mathcal{A}$ is ample and $\mathcal{B}$ is $\overline{f}$-ample, then $\overline{f}^*\mathcal{A} \otimes \varepsilon\mathcal{B}$ is ample on $\mathbb{P}\mathrm{MS}(\mu)$ for small enough ε (see e.g. [Laz04, Section 1.7] for these facts). It thus suffices to show the existence of such an $\overline{f}$-ample bundle.

Our strategy relies on three observations: First, the fibers of $\overline{f}$ are finitely covered by toric varieties. Intuitively, the toric structure stems from rescaling the differentials on subsets of the components of the stable curve (such that the global residue condition is preserved). There we can use the toric Nakai-Kleimann criterion for ampleness.

Proposition 3.1 ([CLS11, Theorem 6.3.13]). *A Cartier divisor D on a proper toric variety X is ample if and only if $D \cdot C > 0$ for every torus-invariant irreducible curve C on X.*

Second, those torus-invariant curves map to a class of curves in $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ that we call relevant curves and that are easy to describe for a given level graph. Third, we can verify on $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ the required positivity by showing the following, using the notation that will be introduced in (9) below. For notation simplicity, we use $\overline{B}$ to denote the moduli space of multi-scale differentials as we did in Section 2.1.

Proposition 3.2. *There is an effective divisor class D such that $\mathcal{L}_{\overline{B}} \otimes \mathcal{O}_{\overline{B}}(-D)$ has positive intersection numbers with all relevant curves in all fibers of $\overline{f}$.*

3.1. **Some line bundles.** Let $\Gamma \in \mathrm{LG}_1(\overline{B})$ be a graph corresponding to a divisor D_Γ in $\overline{B}$. The least common multiple

$$\ell_\Gamma \;=\; \mathrm{lcm}\Big(p_e \;:\; e \in E(\Gamma)\Big)$$

appears frequently in formulas, which is the size of the orbit of the twist group acting on all prong-matchings for Γ. One prominent line bundle is defined by the following sum of boundary divisors

$$\mathcal{L}_{\overline{B}} \;=\; \mathcal{O}_{\overline{B}}\Big(\sum_{\Gamma \in \mathrm{LG}_1(\overline{B})} \ell_\Gamma D_\Gamma\Big)\,. \tag{9}$$

The compactification of the stratum $\overline{B}$, being constructed as the $\mathbb{C}^*$-quotient of the unprojectivized space $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ comes with a tautological bundle $\mathcal{O}_{\overline{B}}(-1)$, whose first Chern class is denoted by ξ.

For inductive arguments we need the following generalization. For $\Gamma \in \mathrm{LG}_L(\overline{B})$, we denote by $\mathrm{i}_\Gamma \colon D_\Gamma \to \overline{B}$ the inclusion that maps the boundary strata into the total space and by $\mathrm{j}_{\Delta,\Gamma} \colon D_\Delta \to D_\Gamma$ the inclusion into an undegeneration. As in [CMZ20b] we denote by $\ell_{\Gamma,i}$ the lcm of the enhancements p_e of the edges e of the two-level undegeneration $\delta_i(\Gamma)$ and let $\ell_\Gamma = \prod_{i=1}^L \ell_{\Gamma,i}$. We now define

$$\mathcal{L}_\Gamma^{[i]} \;=\; \mathcal{O}_{D_\Gamma}\Big(\sum_{\Gamma \overset{[i]}{\rightsquigarrow} \widehat{\Delta}} \ell_{\widehat{\Delta},-i+1} D_{\widehat{\Delta}}\Big) \quad \text{for any} \quad i \in \{0,-1,\dots,-L\}\,, \tag{10}$$

where the sum is over all graphs $\widehat{\Delta} \in \mathrm{LG}_{L+1}(\overline{B})$ that yield divisors in D_Γ by splitting the i-th level.

Generalizing the definition of ξ to the level strata of D_Γ, we define $\xi_\Gamma^{[i]} \in \mathrm{CH}^1(D_\Gamma)$ to be the first Chern class of the tautological bundle at level i on D_Γ.

3.2. **Toric covers.** We start with a description of the fibers of $\overline{f}$ and recall some more details about the construction in [BCGGM2]. Let $(X, \mathbf{z}, \boldsymbol{\eta})$ be a twisted differential consisting of a pointed stable curve $(X, \mathbf{z})$ and a collection $\boldsymbol{\eta} = \{\eta_v\}_{v \in V(\Gamma)}$ of differentials indexed by the vertices of the dual graph Γ of X, satisfying the conditions of a twisted differential compatible with some level structure on Γ, as given in [BCGGM1]. The level structure is not unique, but we can assume that the level structure has a minimal number of levels. The fiber $\overline{F}$ of $\overline{f}$ over (the image in $\overline{M}_{g,n}(\mu)$ of) this twisted differential consists of all multi-scale differentials $(X, \mathbf{z}, \boldsymbol{\omega}, \boldsymbol{\sigma}, \Delta)$ where the collection of differentials $\boldsymbol{\omega}$ is compatible with some enhanced level graph structure Δ on the given dual graph Γ, where $\boldsymbol{\sigma}$ is some prong-matching and where each of the components ω_v of $\boldsymbol{\omega}$ is a multiple of η_v. Recall moreover that two such multi-scale differentials are equivalent if they differ by the action of the level rotation torus T_Δ rescaling the differential level-wise and simultaneously rotating the prong-matchings, see [BCGGM2].

The set of all enhanced level graphs Δ compatible with $\boldsymbol{\eta}$ in this way, with arrows given by undegeneration, forms a directed graph. The terminal elements in this graph, i.e. those with the minimal number of levels, correspond to the irreducible components of $\overline{F}$. By slight abuse of notation, we will denote by Γ such a terminal element.

Consider the action of the 'big' torus $T^{V(\Gamma)}$ rescaling the differentials on each vertex individually. Since the multi-scale differentials are constrained by the global residue condition, which are always of the form that a sum of residues is zero, the orbit of a subtorus $T_\Gamma^P \subset T^{V(\Gamma)}$ preserves the differentials $\boldsymbol{\omega}$ in the fiber $\overline{F}$. The torus T_Γ^P contains the subtorus $T_\Gamma^{\mathrm{np}} \cong (\mathbb{C}^*)^{L(\Gamma)}$ that rescales the differentials level by level (this torus is isogeneous to the level rotation torus T_Γ, but so far we have no prong-matchings taken into account, thus explaining the upper index).

Let $F \subset \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ be the fiber of f over $(X, \mathbf{z}, \boldsymbol{\eta})$. There is a natural map $F \to \overline{F}$, given by passing from the quotient stack (by the automorphism group of the pointed curve and the local factor group K_Γ) to the coarse quotient. Our first goal is to show the following result.

Proposition 3.3. *For each irreducible component F_Γ of F there exists a proper toric variety $\widetilde{F}_\Gamma$ for a torus isogeneous to $T_\Gamma^P/T_\Gamma^{\mathrm{np}}$ that admits a cover $\widetilde{F}_\Gamma \to F_\Gamma$ which is unramified over the open torus orbit.*

The irreducible component F_Γ of the fiber might be called a toric stack. However there are various definitions of that notion and we prefer not entering that discussion. Note that $\overline{F}_\Gamma$ might not be a toric variety due to graph automorphisms, e.g. the quotient of $\mathbb{P}^{46}$ by $\mathbb{Z}/47$ is not even a rational variety ([Swa69]) and such examples can occur in fibers of f for sufficiently large genus.

In order to prove Proposition 3.3 we prove local versions and piece them together as in [Mum83]. The idea was also used in [CMZ20b, Section 4.2] and we use the same notation as there, except that all objects are restricted to a fiber of f and that we need to additionally exhibit a torus action. Let Δ be a degeneration of Γ in the fiber F and let $U(\Delta) \subset F_\Gamma$ be the open substack of multi-scale differentials compatible with undegenerations of Δ.

Lemma 3.4. *There exists a toric variety U_Δ^s for the torus T_Γ^P/T_Γ^{np} that admits an unramified cover $U_\Delta^s \to U_\Delta$ of stacks.*

Proof. We use the cover constructed in [BCGGM2, Section 14] that provides the smooth DM-stack structure of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ by pieces of the simple Dehn space. This cover first provides sufficiently small open neighborhoods in U_Δ with a Tw_Δ-marking, i.e. a marking up to the monodromy in the group Tw_Δ. The smooth charts of the stack are then given by coverings that have a Tw_Δ^s-marking (rather then just a Tw_Δ-marking) and are thus pieces of simple Dehn space. These pieces glue together to the unramified cover $U_\Delta^s \to U_\Delta$. This cover is indeed a smooth complex variety since the simple Dehn space is.

Since the topology in the whole fiber of f is constant we can unwind the definition of the simple Dehn space in [BCGGM2, Section 12] intersected with the fiber of f and obtain the following explicit description. Let $\mathrm{Tw}_\Delta^s = \oplus_{i\in L(\Delta)}\mathrm{Tw}_i^s$ be the level-wise decomposition of the simple twist group and let $T_i = \mathbb{C}/\mathrm{Tw}_i^s$ be the level-wise constituents of the level rotation tori. Then, as a set

$$U_\Delta^s \;=\; \coprod_{\Gamma\rightsquigarrow\Pi\rightsquigarrow\Delta} \left(\mathfrak{W}_{\mathrm{pm}}(\Pi)/\big(\bigoplus_{i\in L(\Pi)} T_i \oplus \bigoplus_{i\in L(\Delta)\setminus L(\Pi)} \mathrm{Tw}_i^s\big)\right),$$

where, with the same deviation of notation compared to the cited sources, $\mathfrak{W}_{\mathrm{pm}}(\Pi)$ denotes the set of all prong-matched differentials compatible with Π in the given fiber of f and with Tw_Π^s-marking. Since all points in $\mathfrak{W}_{\mathrm{pm}}(\Pi)$ are obtained by rescaling $\boldsymbol{\eta}$ and choosing a prong-matching, we find

$$U_\Delta^s \;=\; \coprod_{\Gamma\rightsquigarrow\Pi\rightsquigarrow\Delta} \Big(\bigoplus_{i\in L(\Delta)\setminus L(\Pi)} T_i\Big)\cdot\Big(\coprod_{\boldsymbol{\sigma}} (X, \mathbf{z}, \boldsymbol{\eta}, \boldsymbol{\sigma}, \Pi)\Big), \tag{11}$$

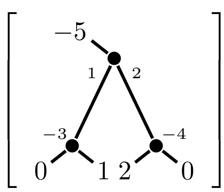
where $\boldsymbol{\sigma}$ runs through representatives of the prong-matchings under the action of $\oplus_{i\in L(\Pi)}T_i$. In particular the torus

$$T^s_{\Delta,\Gamma} \;=\; \bigoplus_{i\in L(\Delta)\setminus L(\Gamma)} T_i$$

acts on U^s_Δ by acting on the stratum for Π via the components in $L(\Delta)\setminus L(\Pi)$. The continuity of this action is clear from the definition of the topology on U^s_Δ. The torus $T^s_{\Delta,\Gamma}$ is a cover of a factor of $T^P_\Gamma/T^{\rm np}_\Gamma$ (in fact isogeneous if Δ is maximally degenerate), since the levels of Δ are obtained by pulling apart the levels of Γ according to the rescalable pieces. The transitivity of the action of $T^s_{\Delta,\Gamma}$ on each irreducible component of U^s_Δ is obvious. □

Proof of Proposition 3.3. Since the claim is for each irreducible component F_Γ of F, we can thus focus on the degenerations of a fixed Γ with minimal number of levels. To ease notation, we keep calling this component F. We take $\widetilde{F}$ to be the normalization of F in the function field of the smallest extension of $K(F)$ that contains field extensions corresponding to $U^s_\Delta \to F$ in Lemma 3.4. Note that there is a finite number of extensions and that these extensions are unramified. Consequently $\widetilde{F}\to F$ is unramified, too. Since each of the U^s_Δ admits an action of a torus $T^s_{\Delta,\Gamma}$ isogeneous to $T^P_\Gamma/T^{\rm np}_\Gamma$, so does $\widetilde{F}$. In fact the fiber product of the $T^s_{\Delta,\Gamma}$ over $T^P_\Gamma/T^{\rm np}_\Gamma$ acts, by the minimality of the field extension. Consequently $\widetilde{F}$ is toric and its properness follows from the properness of F. □

Example 3.5. Consider Γ the 'cherry' graph giving the boundary divisor

$D_\Gamma =$ 

in the stratum with $\mu = (2,1,0,0,-5)$. Since each of the vertices parameterizes a rational curve with three marked points which has unique moduli, the cherry represents a single point in $\overline{M}_{0,5}(\mu)$ (see also [BCGGM2, Example 14.5]). We will describe the fiber F and the toric variety $\widetilde{F}$ in this case.

The residues at all the poles are zero by the Residue Theorem, so that $T^P_\Gamma \cong (\mathbb{C}^*)^2$ rescales independently the vertices on lower level and $T^{\rm np}_\Gamma \cong \mathbb{C}^*$ sits diagonally in T^P_Γ. Let Δ_ℓ (resp. Δ_r) be the slanted cherry graph with the left (resp. right) edge being shorter. We focus on the case $\Delta := \Delta_\ell$. Then, as subgroups of the group $\mathbb{Z}\oplus\mathbb{Z}$ generating the Dehn twists around the left and the right nodes

$$\mathrm{Tw}^s_1 \;=\; \langle (6,0)\rangle, \quad \mathrm{Tw}^s_2 \;=\; \langle (0,3)\rangle, \quad \mathrm{Tw}^s_\Delta \;=\; \mathrm{Tw}^s_1\oplus \mathrm{Tw}^s_2$$

and

$$\mathrm{Tw}_\Delta \;=\; \langle \mathrm{Tw}^s_\Delta, (2,1)\rangle \quad \text{hence} \quad K_\Delta \cong \mathbb{Z}/3\mathbb{Z}\,.$$

In this case $\mathfrak{W}_{\rm pm}(\Gamma) = (\mathbb{C}\times\mathbb{C})/\langle(6,6)\rangle$. The torus $T_1 = \mathbb{C}/6\mathbb{Z}$ acts diagonally on this space and the discrete group Tw^s_2 acts effectively on the second factors. In this decomposition

$$U^s_\Delta \;=\; U^s_\Delta(\Gamma) \amalg U^s_\Delta(\Delta)\,,$$

and we have thus identified the first subset $U^s_\Delta(\Gamma) = \mathfrak{W}_{\rm pm}(\Gamma)/(T_1 \oplus {\rm Tw}^s_2)$. On the other hand,

$$\mathfrak{W}_{\rm pm}(\Delta) \;=\; (\mathbb{C}\times\mathbb{C})/\langle(6,6),(0,3)\rangle \quad\text{and}\quad U^s_\Delta(\Delta) = \mathfrak{W}_{\rm pm}(\Delta)/T_1 \oplus T_2\,,$$

where $T_1 = \mathbb{C}/6\mathbb{Z}$ acts diagonally as in the preceding case and $T_2 = \mathbb{C}/3\mathbb{Z}$ acts on the second factor. Obviously $T_1 \oplus T_2$ acts faithfully and transitively and $U^s_\Delta(\Delta)$ is a single point. The group K_Δ acts faithfully on the first subset while fixing the second and thus produces the non-trivial quotient stack structure of U_Δ at the image point of the left slanted cherry.

The more useful description of this decomposition of U^s_Δ is (11). Since T_1 acts transitively on the prong-matchings for both subsets, the decomposition into prong-matchings representatives is reduced to a single factor, a single prong-matching equivalence class. The set U^s_Δ is thus a toric variety for $T_2 = (0\times\mathbb{C})/(0\times 3\mathbb{Z})$, which is a triple cover of $T^P_\Gamma/T^{\rm np}_\Gamma$, where $T^P_\Gamma = (\mathbb{C}\times\mathbb{C})/(\mathbb{Z}\times\mathbb{Z})$ and $T^{\rm np}_\Gamma$ is the diagonal in T^P_Γ.

The same description holds for Δ_r exchanging the role of the prong numbers $p_1 = 2$ and $p_2 = 3$ everywhere.

Consequently, the full fiber F consists a complex plane U_{Δ_ℓ} with orbifold order three at the origin, glued via $z \mapsto 1/z$ to a complex plane U_{Δ_r} with orbifold order two at the origin. The cover $\widetilde{F} \to F$ is a cyclic cover of degree six, fully ramified over the origin and ∞, which is the smallest cover that dominates the cyclic cover of order three ramified at 0 and that of order two ramified at ∞. The fiber product of T_2 and the corresponding torus for the right slanted cherry $T_1 = (\mathbb{C}\times 0)/(2\mathbb{Z}\times 0)$ over $T^P_\Gamma/T^{\rm np}_\Gamma$ admits an isogeny of degree six to $T^P_\Gamma/T^{\rm np}_\Gamma$ and acts on $\widetilde{F}$ as requested.

3.3. **Relevant curves.** We introduce the notion of *relevant curves* in $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ constructed as follows. In the first step take a boundary stratum D_Δ, say with $\Delta \in {\rm LG}_L(\overline{B})$ that has the following two features.

First, one level i whose vertices $V^{[i]}$ can be partitioned into two sets $V^{[i]}_A$ and $V^{[i]}_B$ with the following property. There exists a codimension-one degeneration $\Delta^A \in {\rm LG}_{L+1}(\overline{B})$ of Δ such that the i-th level is split and the vertices in A go down while those in B stay up, and vice versa a codimension-one degeneration $\Delta^B \in {\rm LG}_{L+1}(\overline{B})$ of Δ where those in B go down and those in A stay up. (It may happen that the two degenerations produce abstractly isomorphic graphs, see the rhombus graph in Example 3.9 below. Note that just being able to put A down while keeping B up (without the converse) is not a sufficient criterion, see e.g. the zig-zag graph in [CMZ20b, Figure 2].)

Second, there is a unique level i with such a splitting and there is no finer partition of the vertices at level i that can be moved up and down independently (this also justifies that the vertices involved in the splitting into A and B are suppressed in the notation of relevant curves).

In the second step we define the relevant curve $C_{\Delta,i}$ inside D_Δ given by specifying a point class on the generalized stratum of each level different from i, and a point class on the generalized strata corresponding to the partitions A and B (these are the generalized strata at level i of Δ^A and Δ^B respectively). Note that a boundary stratum can be disconnected due to prong-matching equivalence classes (see e.g. [CMZ20b, Section 3]), but this second step pins down a component the relevant curve lies in. We do no record this in the notation, since the intersection numbers below do not depend on the component of the boundary stratum.

Obviously relevant curves are contained in the fibers of f.

Lemma 3.6. *Let C be a torus-invariant irreducible curve in the cover $\widetilde{F}$ of any fiber F of f. Then the image of C in $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is a relevant curve.*

The converse statement also holds by the same argument, i.e. relevant curves are torus-invariant, but it is not needed in the sequel.

Proof. Recall that we covered F by the images of the sets U^s_Δ. For a torus-invariant curve C, the generic point of C thus lies in the preimage of some open set U^s_Δ, and hence C is given by the closure of certain one-dimensional subtorus T_1 of $T := T^P_\Gamma/T^{\mathrm{np}}_\Gamma$.

First suppose that T_1 acts on at least two levels i and j non-trivially. Decompose the vertices on level i into three subsets A_i, B_i and C_i, where T_1 scales the vertices in A_i down at its parameter $t = 0$, scales the vertices in B_i down at $t = \infty$, and does not scale the vertices in C_i. Note that A_i and B_i are non-empty and C_i can possibly be empty. In the same way we decompose the vertices on level j into three subsets A_j, B_j and C_j. Then the subtorus of T that rescales A_i with t and A_j with t^{-1} (and does nothing to the other subsets) exhibits C as not T-invariant, which contradicts the assumption.

Next suppose that the action of T_1 is non-trivial on level i only and partitions the vertices of that level into three non-empty subsets, the set A that goes down at $0 \in \overline{T}_1$, the set B that goes down at $\infty \in \overline{T}_1$, and the rest S. Then the subtorus of T that fixes $A \cup B$ and rescales the vertices in S diagonally exhibits C as not T-invariant, leading to the same contradiction.

Therefore, T_1 acts non-trivially only on a single level i and decomposes the vertices on level i into two subsets A and B which are obtained by considering the limits to 0 and to $\infty \in \overline{T}_1$. Note that this argument justifies the uniqueness and minimality in the second part of the first step defining relevant curves, while the second step in the definition simply cuts down the dimension to one (i.e. to a curve). $\square$

Using Proposition 3.2 we can now complete:

Proof of Theorem 1.1. The bundle $\mathcal{B} = \mathcal{L}_{\overline{B}} \otimes \mathcal{O}_{\overline{B}}(-D)$ from Proposition 3.2 descends to a bundle $\overline{\mathcal{B}}$ on $\mathbb{P}\mathrm{MS}(\mu)$ since the boundary divisors (and thus $\mathcal{L}_{\overline{B}}$) are invariant under the local isomorphism groups. We claim that $\overline{\mathcal{B}}$ is $\overline{f}$-ample. By definition we need to show that the restriction of $\overline{\mathcal{B}}$ to any fiber of $\overline{f}$ is ample. For this it suffices to prove the ampleness of the pullback via the finite covering $\widetilde{F} \to F \to \overline{F}$ given by Proposition 3.3, which implies that we can check ampleness via the toric criterion in Proposition 3.1. Namely, we need to check the positivity of the pullback of $\overline{\mathcal{B}}$ to $\widetilde{F}$ on any torus-invariant curve. Then by Lemma 3.6 and push-pull it suffices to check the positivity of $\mathcal{B}$ on any relevant curve. We have thus reduced the claim to that of Proposition 3.2. $\square$

3.4. **The proof of positivity.** It remains to show Proposition 3.2. Recall from the description of the relevant curves $C_{\Delta,i}$ above that among the boundary divisors that do not contain $C_{\Delta,i}$, there are precisely two divisors D_{Γ^A} and D_{Γ^B} with non-zero (hence positive) intersection numbers with $C_{\Delta,i}$, namely the ones containing D_{Δ^A} and D_{Δ^B}, respectively.

Lemma 3.7. *Let $C_{\Delta,i}$ be a relevant curve. Then*

$$\big(\xi_\Delta^{[i]} + \mathrm{c}_1(\mathcal{L}_\Delta^{[i]})\big) \cdot [C_{\Delta,i}] > 0 \quad \textit{and} \quad (\xi_\Delta^{[i]}) \cdot [C_{\Delta,i}] < 0\,.$$

Proof. For the first statement we express $\xi_\Delta^{[i]}$ using [CMZ20b, Proposition 8.2] as a positive ψ-class contribution and a negative boundary contribution, consisting of certain summands that also appear in $\mathcal{L}_\Delta^{[i]}$. Since ψ-classes are pullbacks from $\overline{\mathcal{M}}_{g,n}$, they have zero intersection numbers with contracted curves. The boundary divisors in $\mathcal{L}_\Delta^{[i]}$ do not contain $C_{\Delta,i}$, so their intersection numbers with $C_{\Delta,i}$ are non-negative. More precisely, the boundary terms in [CMZ20b, Proposition 8.2] are (for a chosen leg x specifying the ψ-class) those where x goes down in the splitting. This set contains exactly one of D_{Δ^A} and D_{Δ^B} since in the definition of relevant curves the vertices in the split level are partitioned into A and B. So the contribution is positive, as claimed.

The second statement follows from the same relation, since the boundary terms in this relation do not contain C_{Δ_i}, and since one of D_{Δ^A} and D_{Δ^B} appears in the relation (with negative sign). □

Lemma 3.8. *Let $C_{\Delta,i}$ be a relevant curve with $\Delta \in \mathrm{LG}_L(\overline{B})$. Then*

$$\mathrm{c}_1(\mathcal{L}_{\overline{B}}) \cdot [C_{\Delta,i}] > 0 \quad \textit{if} \quad i \in \{0, -L\}$$

and $\mathrm{c}_1(\mathcal{L}_{\overline{B}}) \cdot [C_{\Delta,i}] = 0$ *otherwise.*

Proof. We denote the successive undegenerations of Δ keeping the top j levels by $\Delta_j := \delta_{1,\dots,j}(\Delta)$ and $\Delta_0 := \overline{B}$. In particular $\Delta_L = \Delta$. Define moreover the successive pullbacks of $\mathcal{L}_{\overline{B}}$ to be $\mathcal{E}_j := \mathrm{j}^*_{\Delta_j,\Delta_{j-1}}(\mathcal{E}_{j-1})$, where $\mathcal{E}_0 := \mathcal{L}_{\overline{B}}$. In particular $\mathrm{c}_1(\mathcal{E}_L) = \mathrm{c}_1(\mathcal{L}_{\overline{B}}) \cdot [D_\Delta]$. Applying [CMZ20b, Lemma 7.6] successively we find that $\mathrm{c}_1(\mathcal{E}_L) = \mathrm{c}_1(\mathcal{L}_\Delta^{[-L]}) + \xi_\Delta^{[-L]} - \xi_\Delta^{[0]}$. The desired claim thus follows from Lemma 3.7. □

Finally recall from [CMZ20b, Theorem 7.1] that the normal bundle of a boundary divisor is given by

$$\mathrm{c}_1(\mathcal{N}_\Gamma) \;=\; \frac{1}{\ell_\Gamma}\big(-\xi_\Gamma^\top - \mathrm{c}_1(\mathcal{L}_\Gamma^\top) + \xi_\Gamma^\perp\big) \quad \text{in} \quad \mathrm{CH}^1(D_\Gamma)\,. \tag{12}$$

More generally, the normal bundle of a codimension-one degeneration of graphs, say with $\delta^{\complement}_{-i+1}(\Gamma) = \Pi$ is given in loc. cit. by

$$\mathrm{c}_1(\mathcal{N}_{\Gamma,\Pi}) \;=\; \frac{1}{\ell_{\Gamma,-i+1}}\big(-\xi_\Gamma^{[i]} - \mathrm{c}_1(\mathcal{L}_\Gamma^{[i]}) + \xi_\Gamma^{[i-1]}\big) \quad \text{in} \quad \mathrm{CH}^1(D_\Gamma)\,. \tag{13}$$

Proof of Proposition 3.2. We start by numbering the (non-horizontal) boundary divisors $D_1, D_2, \dots$ of $\overline{B}$ in such a way that whenever two boundary divisors intersect, the one with smaller index will be obtained as undegeneration keeping the top level passage. In symbols, if $\Gamma \in \mathrm{LG}_2(\overline{B})$ is a graph corresponding to a boundary stratum in $D_i \cap D_j$ with $i < j$, then $D_{\delta_1(\Gamma)} = D_i$ and $D_{\delta_2(\Gamma)} = D_j$. This is possible because of [CMZ20b, Proposition 5.1], or equivalently it is a total order refining the partial order defined in [CMZ20a, Proposition 3.5].

Consider now the set $\mathfrak{C}$ of relevant curves for $\overline{B}$, which we write as the disjoint union of $\mathfrak{C}_E$ and $\mathfrak{C}_H$. The first set $\mathfrak{C}_E$, 'easy' to deal with, consists of relevant curves $C_{\Delta,i}$ such that $i = 0$ or $\Delta \in \mathrm{LG}_i(\overline{B})$, i.e. the split level is the top or bottom level of Δ. 'Hard' to deal with is $\mathfrak{C}_H = \mathfrak{C} \setminus \mathfrak{C}_E$, i.e. the split level is strictly in between.

Lemma 3.8 shows that $\mathcal{L}_{\overline{B}}$ intersects all relevant curves non-negatively, and intersects the easy ones positively. The strategy is to create positive intersections with curves in $\mathfrak{C}_H$ without destroying the positivity we already have. For this purpose we next decompose $\mathfrak{C}_H$ further.

To a relevant curve $C_{\Delta,i} \in \mathfrak{C}_H$ with $\Delta \in \mathrm{LG}_L(\overline{B})$ and $0 > i > -L$, we can associate a three-level graph $\widetilde{\Delta}$ by keeping only the levels right above and below the critical level i. In symbols $\widetilde{\Delta} = \delta_{-i,-i+1}(\Delta) \in \mathrm{LG}_2(\overline{B})$. We define $\mathfrak{C}_H^{a,b}$, for $a < b$, to be the set of relevant curves $C_{\Delta,i}$ whose associated graph $\widetilde{\Delta}$ labels a component of the intersection $D_a \cap D_b$. In other words, if D_Δ is a component of $D_{k_1} \cap \cdots \cap D_{k_L}$ with $k_1 < \cdots < k_L$, then $k_{-i} = a$ and $k_{-i+1} = b$.

In what follows we only need to consider the boundary divisors D_a where $\mathfrak{C}_H^{a,b}$ is non-empty for some $b > a$. For notation simplicity we still label them in increasing order as $D_1, D_2, \ldots$. We start by considering $\mathfrak{C}_H^{1,b}$ for $b > 1$ and the line bundle $\mathcal{L}_{\varepsilon_1} = \mathcal{L}_{\overline{B}} \otimes \mathcal{O}(-\varepsilon_1 D_1)$ for $\varepsilon_1 > 0$. We claim that for ε_1 small enough

$$\mathrm{c}_1(\mathcal{L}_{\varepsilon_1}) \cdot [C] > 0 \quad \text{for} \quad C \in \mathfrak{C}_E \cup \bigcup_{b>1} \mathfrak{C}_H^{1,b} \tag{14}$$

and $\mathrm{c}_1(\mathcal{L}_{\varepsilon_1}) \cdot [C] \geq 0$ for the remaining relevant curves. First, to justify the last part it suffices to show that $[D_1] \cdot [C] = 0$ for $C \in \mathfrak{C}_H^{a,b}$ with $a > 1$. If D_1 contains $C = C_{\Delta,i}$, then $1 < a$ implies that D_1 can be obtained as an undegeneration keeping the level passage one or more above level i. Namely, in the above notation D_Δ is a component of $D_{k_1} \cap \cdots \cap D_{k_L}$ with $k_1 = 1$, $k_{-i} = a$, $k_{-i+1} = b$ and $-i > 1$. Pull back the normal bundle $\mathcal{N}_{D_1}$ successively to $D_1 \cap D_{k_2} \cap \cdots \cap D_{k_j}$ with j varying from 2 to L, and apply [CMZ20b, Corollary 7.7]. We obtain that $[D_1] \cdot [C]$ is equal to the degree of $\mathcal{N}_{\Delta,\delta_1^{\complement}(\Delta)}$ on C, which is zero by using (13) (with $i = 0$ therein and noting that a relevant curve obtained by splitting level i has zero intersection with $\mathrm{c}_1(\mathcal{L}_\Delta^{[j]})$ and $\xi_\Delta^{[j]}$ for $j \neq i$). So the only way the claim can fail is that D_1 and $C = C_{\Delta,i}$ intersect in one of the two graphs Δ^A or Δ^B. Undegenerating the levels away from those adjacent to level i of Δ^A or Δ^B, we obtain a four-level graph Δ' such that $\delta_1(\Delta') = a$, $\delta_2(\Delta') = 1$ and $\delta_3(\Delta) = b$. This graph would correspond to an intersection of D_a and D_1 with D_a on top. By the initial choice of the ordering of the divisors, this is not possible. Second, for $C = C_{\Delta,i} \in \mathfrak{C}_H^{1,b}$, in the above notation it means $i = -1$. Pull back the normal bundle $\mathcal{N}_{D_1}$ successively to $D_{k_1} \cap D_{k_2} \cap \cdots \cap D_{k_j}$ with $k_1 = 1$, $k_2 = b$ and j varying from 2 to L, and apply [CMZ20b, Corollary 7.7]. We thus obtain that

$$-\varepsilon_1 [D_1] \cdot [C_{\Delta,-1}] \;=\; -\varepsilon_1 \deg\left(\mathcal{N}_{\Delta,\delta_1^{\complement}(\Delta)}|_C\right) \;=\; -\frac{\varepsilon_1}{\ell_{\Gamma,1}} \xi_\Delta^{[-1]} \cdot [C_{\Delta,-1}] > 0 \tag{15}$$

where the last equality and inequality follow from (13) and Lemma 3.7 respectively. Third, the positivity for $C \in \mathfrak{C}_E$ already established is not destroyed for ε_1 small enough.

We next consider $\mathfrak{C}_H^{2,b}$ for $b > 2$ and the line bundle $\mathcal{L}_{\varepsilon_1,\varepsilon_2} = \mathcal{L}_{\varepsilon_1} \otimes \mathcal{O}(-\varepsilon_2 D_2)$. Again, we claim that for ε_2 small enough

$$\mathrm{c}_1(\mathcal{L}_{\varepsilon_1,\varepsilon_2}) \cdot [C] > 0 \quad \text{for} \quad C \in \mathfrak{C}_E \cup \bigcup_{\substack{a \in \{1,2\} \\ b > a}} \mathfrak{C}_H^{a,b} \tag{16}$$

and $\mathrm{c}_1(\mathcal{L}_{\varepsilon_1,\varepsilon_2}) \cdot [C] \geq 0$ for the remaining relevant curves. As in the previous step, the claim about the remaining curves follows from the ordering of the divisors D_i.

The claim about $\mathfrak{C}_H^{2,b}$ follows from the description of the normal bundle, and the positivity of the intersection pairing with curves in $\mathfrak{C}_E$ and $\mathfrak{C}_H^{1,b}$ is not destroyed for ε_2 small enough.

Iteratively we can define $\mathcal{L}_{\varepsilon_1,\dots,\varepsilon_j} = \mathcal{L}_{\varepsilon_1,\dots,\varepsilon_{j-1}} \otimes \mathcal{O}(-\varepsilon_j D_j)$ with ε_j small enough until at the last step the resulting line bundle class intersects all relevant curves positively. $\square$

Example 3.9. The multi-scale space $\overline{B} = \mathbb{P}\Xi\mathcal{M}_{1,4}(2,0,0,-2)$ has six relevant curves. Consider first the set $\mathfrak{C}_E$ of 'easy' curves. There are four curves in this set, which are described in Figure 2. The first two are supported on divisors, so we need to impose an extra codimension-one condition, for example a ψ-decoration, in order to make them into curve classes, while the last two are honest one-dimensional boundary strata. In all of these cases, the split level is the bottom one. Hence by Lemma 3.8 they intersect positively with $\mathrm{c}_1(\mathcal{L}_{\overline{B}})$. One can check that the intersection numbers of $\mathrm{c}_1(\mathcal{L}_{\overline{B}})$ with these four curves are given by

$$\mathrm{c}_1(\mathcal{L}_{\overline{B}}) \cdot [C_{\Delta_1,-1}] = \frac{1}{8}\,, \quad \mathrm{c}_1(\mathcal{L}_{\overline{B}}) \cdot [C_{\Delta_2,-1}] = \frac{1}{2}\,,$$
$$\mathrm{c}_1(\mathcal{L}_{\overline{B}}) \cdot [C_{\Delta_3,-2}] = \frac{1}{2}\,, \quad \mathrm{c}_1(\mathcal{L}_{\overline{B}}) \cdot [C_{\Delta_4,-2}] = 1\,.$$

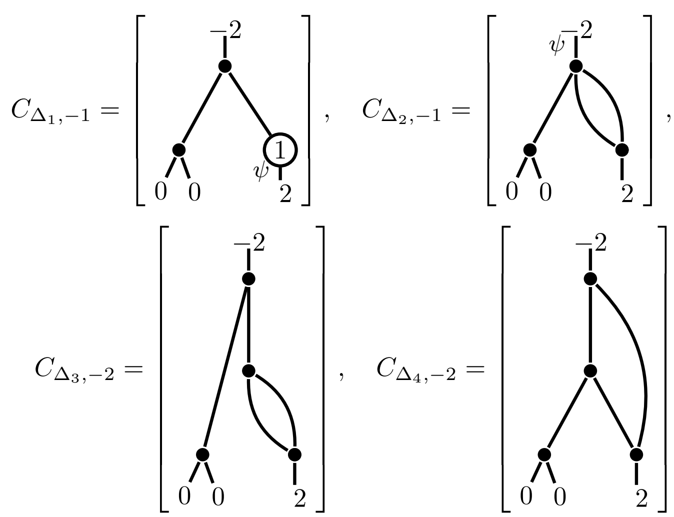


FIGURE 2. The set $\mathfrak{C}_E$ of 'easy' relevant curves in the space $\mathbb{P}\Xi\mathcal{M}_{1,4}(2,0,0,-2)$ which intersect positively with $\mathcal{L}_{\overline{B}}$.

There are two 'hard' relevant curves, given by boundary strata defined by three-level graphs where the split level is at level -1 (see Figure 3). The first curve $C_{\Delta_5,-1}$ is of cherry-banana type. The other curve $C_{\Delta_6,-1}$ is of rhombus type. By Lemma 3.8, the intersection numbers of $\mathrm{c}_1(\mathcal{L}_{\overline{B}})$ with these two curves are zero. Let D_1 be the divisor of cherry type given by the undegeneration of $C_{\Delta_5,-1}$ keeping the first level passage. Let D_2 be the divisor of banana type given by the undegeneration of $C_{\Delta_6,-1}$ keeping the first level passage. One can check that

$$[D_1] \cdot [C_{\Delta_5,-1}] = -\frac{1}{2}\,, \quad [D_2] \cdot [C_{\Delta_6,-1}] = -1\,.$$

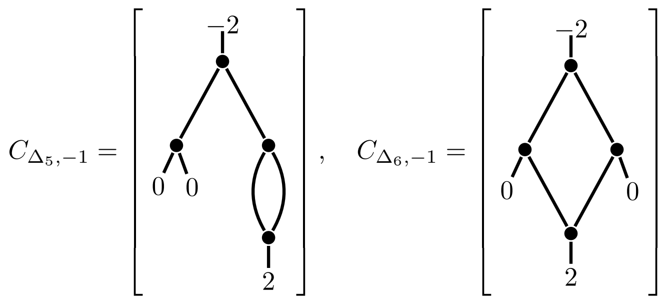


FIGURE 3. The set $\mathfrak{C}_H$ of 'hard' relevant curves in the space $\mathbb{P}\Xi\mathcal{M}_{1,4}(2,0,0,-2)$ which have zero intersection with $\mathcal{L}_{\overline{B}}$.

One can also see that besides $C_{\Delta_5,-1}$ among all the relevant curves D_1 intersects non-trivially only $C_{\Delta_1,-1}$ which evaluates to $-1/8$, while D_2 intersects trivially all the relevant curves apart from $C_{\Delta_6,-1}$. Hence in this case the line bundle $c_1(\mathcal{L}_{\overline{B}}) - \varepsilon_1[D_1] - \varepsilon_2[D_2]$ with $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ is f-ample.

In this example, one can also compute (with the help of `diffstrata`) the cone of f-ample divisors. It is a 13-dimensional polyhedron defined as the convex hull of 1 vertex, 9 rays, and 5 lines.

4. The age of automorphisms

This section prepares for determining non-canonical singularites in $\mathbb{P}\mathrm{MS}(\mu)$ in the subsequent Section 5. The first step is to determine ages of automorphisms, since we want to apply a variant of the Reid–Tai criterion for canonical singularities ([Rei87], [Tai82]). We give age estimates in two situations, first for interior points of strata allowing permuted marked points and second at the boundary, restricting there to fixed marked points since this is our main goal.

From now on let $B = \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ be the moduli space of Abelian differentials of type $\mu \in \mathbb{Z}^n$, so of possibly meromorphic type, and $\overline{B} = \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ be its compactification by multi-scale differentials. For the subsequent propositions we split the set of marked points $\mathbf{z}$ of a multi-scale differential into the set Z of zeros ($m_i \geq 0$) and into the set P of poles ($m_i < 0$).

First of all we consider automorphisms of Abelian differentials with possibly unlabeled marked points. Recall that the local deformation space of an Abelian differential (X, ω, Z, P) can be identified with $H^1(X \setminus P, Z; \mathbb{C})$. The tangent space to the associated projectivized stratum is thus naturally identified with the affine space $\mathbb{A}_{\mathrm{rel}}(X)$ introduced in (4), where in this case we only deal with a one-level graph consisting of a single vertex. When computing the age of an automorphism, it will be on this affine space throughout. We write $\Omega\mathcal{M}_g(\mu)$ for the spaces with unlabeled zeros and poles.

Proposition 4.1. *Let (X, Z, P) be a (stable) pointed smooth curve with an automorphism τ of order $k \geq 2$, fixing zeros and poles setwise (but not necessarily pointwise) and fixing projectively an Abelian differential ω of type μ with the zeros Z and poles P. Let ζ be any primitive k-th root of unity, and let $\zeta^{a'_1}, \ldots, \zeta^{a'_d}$ be the eigenvalues of the induced action on $\mathbb{A}_{\mathrm{rel}}(X)$, where $0 \leq a'_i < k$ and $d = \dim(B)$.*

Case	Stratum	order	eigenvalues	age $\geq$
(H)	hyperelliptic differentials	$k=2$		0
(1)	$\mathbb{P}\Omega\mathcal{M}_1(\{0,0\})$	$k=2$	$1,1,-1$	$1/2$
(2)	$\mathbb{P}\Omega\mathcal{M}_0(\{m,m,m\},-3m-2)$	$k=3$	ζ_3,ζ_3^2	$1/3$
(3)	$\mathbb{P}\Omega\mathcal{M}_2(m,2-m),\ 1\neq m\equiv 1 \bmod 3$	$k=3$	$\zeta_3,\zeta_3,\zeta_3^2,\zeta_3^2$	$2/3$
(4)	$\mathbb{P}\Omega\mathcal{M}_1(0)$	$k=3$	ζ_3,ζ_3^2	$1/3$
(5)	$\mathbb{P}\Omega\mathcal{M}_1(m,-m),\ 0\neq m\equiv 0 \bmod 3$	$k=3$	ζ_3,ζ_3^2	$1/3$
(6)	$\mathbb{P}\Omega\mathcal{M}_1(\{0,0,0\})$	$k=3$	$\zeta_3,\zeta_3,\zeta_3^2,\zeta_3^2$	$2/3$
(7)	$\mathbb{P}\Omega\mathcal{M}_1(\{m,m,m\},-3m),\ m\neq 0$	$k=3$	$\zeta_3,\zeta_3,\zeta_3^2,\zeta_3^2$	$2/3$
(8)	$\mathbb{P}\Omega\mathcal{M}_0(\{m,m,m,m\},-4m-2)$	$k=4$	$i,-1,i^3$	$3/4$
(9)	$\mathbb{P}\Omega\mathcal{M}_1(0)$	$k=4$	i,i^3	$1/2$
(10)	$\mathbb{P}\Omega\mathcal{M}_1(m,-m),\ 0\neq m$ even	$k=4$	i,i^3	$1/2$
(11)	$\mathbb{P}\Omega\mathcal{M}_1(\{0,0\})$	$k=4$	$i,-1,i^3$	$3/4$
(12)	$\mathbb{P}\Omega\mathcal{M}_1(\{m,m\},-2m),\ 0\neq m$ even	$k=4$	$i,-1,i^3$	$3/4$
(13)	$\mathbb{P}\Omega\mathcal{M}_1(0)$	$k=6$	ζ_6,ζ_6^5	$1/3$

FIGURE 4. Automorphisms with age <1. The column of eigenvalues corresponds to the induced action on the unprojectivized chart in $H^1(X\setminus P,Z;\mathbb{C})$. Here m can be possibly negative.

Then

$$\operatorname{age}(\tau|_{\mathbb{A}_{\mathrm{rel}}(X)}) := \sum_{i=1}^{d}\frac{a_i'}{k} \geq 1,$$

except for the cases listed in Figure 4.

Consider now the space $\overline{B}$. Recall from Section 2.2 the local coordinate system near a multi-scale differential $(X,\boldsymbol{\omega},\mathbf{z},\boldsymbol{\sigma})\in\overline{B}$ compatible with the enhanced level graph Γ, in particular the decomposition of the coordinates in (5). Moreover there we explained that automorphisms of multi-scale differentials are only well defined on the quotient of the affine space $\mathbb{A}=\mathbb{A}_{\mathrm{rel}}(X)\times\mathbb{C}_{\mathrm{hor}}\times\mathbb{C}_{\mathrm{lev}}$ by the group K_Γ. On the other hand the age of an automorphism is defined only for a linear action. We abuse this definition for $\boldsymbol{\tau}\in\operatorname{Aut}(X,\boldsymbol{\omega},\mathbf{z},\boldsymbol{\sigma})$ and say that $\operatorname{age}(\boldsymbol{\tau})\geq C$ if each lift of $\boldsymbol{\tau}$ to an automorphism of $\mathbb{A}$ has age $\geq C$. In particular, if the induced action of $\boldsymbol{\tau}$ on $\mathbb{A}_{\mathrm{rel}}(X)$ or on $\mathbb{A}_{\mathrm{rel}}(X)\times\mathbb{C}_{\mathrm{hor}}$ has age $\geq C$, then $\operatorname{age}(\boldsymbol{\tau})\geq C$.

Proposition 4.2. *Let Γ be a level graph representing a boundary stratum in $\overline{B}$ and let $\boldsymbol{\tau}=(\tau_{(-i)})$ be an automorphism of a multi-scale differential compatible with Γ, fixing the labeled points. Suppose moreover that $\boldsymbol{\tau}$ fixes a vertex v. Then* $\operatorname{age}(\boldsymbol{\tau})\geq 1$, *if v does not belong to the lists in Figure 4 and Figure 5.*

Case	Stratum	order	eigenvalues	age $\geq$
(RH)	hyperelliptic differentials	$k=2$		0
$(R1)$	$\mathbb{P}\Omega\mathcal{M}_0^{\mathfrak{R}}(3m_1+m_2-2, -m_2, \{-m_1\}^3)$, $m_2 \not\equiv 1 \bmod 3$, $\mathfrak{R} = \{r_2 = 0\}$ or $\mathfrak{R} = \{r_2 = 0,\, r_3+r_4+r_5 = 0\}$	$k=3$	ζ_3, ζ_3^2	1/3
$(R2)$	$\mathbb{P}\Omega\mathcal{M}_0^{\mathfrak{R}}(\{m_1+m_2-1\}^3, -3m_1, 1-3m_2)$, $\mathfrak{R} = \{r_4 = 0\}$ or $\mathfrak{R} = \{r_5 = 0\}$ or $\mathfrak{R} = \{r_4 = r_5 = 0\}$	$k=3$	ζ_3, ζ_3^2	1/3
$(R3)$	$\mathbb{P}\Omega\mathcal{M}_{2,4}^{\mathfrak{R}}(\sum_{i=1}^3 m_i + 2, -m_1, -m_2, -m_3)$, $m_i \not\equiv 1 \bmod 3,\ \sum_{i=1}^3 m_i \not\equiv 0 \bmod 3$, $\mathfrak{R} = \{r_2 = r_3 = 0\}$ or $\mathfrak{R} = \{r_2 = r_3 = r_4 = 0\}$	$k=3$	$\zeta_3, \zeta_3, \zeta_3^2, \zeta_3^2$	2/3
$(R4)$	$\mathbb{P}\Omega\mathcal{M}_{2,4}^{\mathfrak{R}}(m_1+m_2+2, -m_1, -m_2)$, $m_i \not\equiv 1 \bmod 3,\ m_1+m_2 \not\equiv 0 \bmod 3$, $\mathfrak{R} = \{r_2 = 0\}$ or $\mathfrak{R} = \{r_2 = r_3 = 0\}$	$k=3$	$\zeta_3, \zeta_3, \zeta_3^2, \zeta_3^2$	2/3
$(R5)$	$\mathbb{P}\Omega\mathcal{M}_{1,3}^{\mathfrak{R}}(m_1+m_2, -m_1, -m_2)$, $m_i \not\equiv 1 \bmod 3,\ m_1+m_2 \not\equiv 2 \bmod 3$, $\mathfrak{R} = \{r_2 = 0\}$ or $\mathfrak{R} = \{r_2 = r_3 = 0\}$	$k=3$	ζ_3, ζ_3^2	1/3
$(R6)$	$\mathbb{P}\Omega\mathcal{M}_1^{\mathfrak{R}}(\{m_1+m_2\}^2, -2m_1, -2m_2)$, m_1+m_2 even, $\mathfrak{R} = \{r_3 = 0\}$ or $\mathfrak{R} = \{r_3 = r_4 = 0\}$	$k=4$	$i, -1, i^3$	3/4
$(R7)$	$\mathbb{P}\Omega\mathcal{M}_1^{\mathfrak{R}}(4m_1+2m_2, \{-2m_1\}^2, -2m_2)$, $\mathfrak{R} = \{r_4 = 0\}$ or $\mathfrak{R} = \{r_2 + r_3 = 0\}$	$k=4$	$i, -1, i^3$	3/4

FIGURE 5. Vertices with residue conditions that can yield $\operatorname{age}(\boldsymbol{\tau}) < 1$. Here the m_i are always positive. Each r_i denotes the residue of the i-th entry in the signature. Alternative versions of the GRC are equivalent by the Residue Theorem and relabeling the poles.

Remark 4.3. Let X be a hyperelliptic curve of genus g with f_z fixed zeros, f_p fixed poles, c_z conjugate pairs of zeros and c_p conjugate pairs of poles, under the hyperelliptic involution τ. It is easy to check that the eigenvalue decomposition of the τ-action on the (unprojectivized) relative periods (if unconstrained by the GRC) is

$$(-1)^{2g}\,,\ (+1)^{c_z+f_z-1}\,,\ (-1)^{c_z}\,,\ (+1)^{c_p+f_p-1}\,,\ (-1)^{c_p}$$

where the fourth term is empty if $c_p + f_p = 0$ (i.e., when there is no pole). In particular, if (the projectivized) $\operatorname{age}(\tau) < 1$ then $c_z + f_z \leq 2$ and $c_p + f_p \leq 2$, and not both of these are equal to 2. For later use we also need to consider hyperelliptic involutions with GRC constraints. In particular if $\operatorname{age}(\tau) = 0$, then every fixed pole of X is constrained by the GRC to have zero residue and every conjugate pair of poles of X is constrained by the GRC to have the sum of the residues equal to zero.

Remark 4.4. Conversely, all the cases listed in the tables can be realized via canonical covers of k-differentials on $X/\langle\tau\rangle$, and all the congruence conditions of the signatures are necessary as well. We go over one case in each table and leave the others for the reader to verify. For example for $\mathbb{P}\Omega\mathcal{M}_0(\{m,m,m,m\},-4m-2)$ in Case (8), take a quartic differential of signature $(4m,-4m-5,-3)$ in $\mathbb{P}^1$ and pull it back via the canonical quartic cover totally ramified at the last two singularities, where the second ramification point over the pole of order three becomes an unmarked ordinary point. Next for the cases in Figure 5, first note that if ω is not a $\boldsymbol{\tau}$-invariant form, then any $\boldsymbol{\tau}$-fixed pole of ω must have zero residue. In addition, if there is a totally ramified point of multiplicity k under the $\boldsymbol{\tau}$-action, then ω is an invariant form if and only if the singularity order at the ramification point is $\equiv k-1 \bmod k$. Using these, consider Case (R1) as an example. One can pull back a cubic differential of signature $(3m_1+m_2-4,-m_2-2,-3m_1)$ on $\mathbb{P}^1$ via the canonical triple cover totally ramified at the first two singularities, whose cubic root thus gives the desired ω satisfying the residue condition $\mathfrak{R}$ as well.

In order to prove the above propositions, we make some preparation first. Let (X,ω,Z,P) be an Abelian differential and τ be an automorphism of order k, either in the context of Proposition 4.1 or as the restriction of $\boldsymbol{\tau}$ from Proposition 4.2 to the surface at the vertex v. We fix a primitive k-th root of unity $\zeta=\zeta_k$ and $a\in\{0,\dots,k-1\}$ such that

$$\tau^*\omega \;=\; \zeta^a\omega\,. \tag{17}$$

Let γ be a homology class such that $\tau_*\gamma=\zeta^{a_i}\gamma$. Then

$$\zeta^{a_i}\int_\gamma\omega \;=\; \int_{\tau_*\gamma}\omega \;=\; \int_\gamma\tau^*\omega \;=\; \zeta^a\int_\gamma\omega\,.$$

Therefore, the eigen-period $\int_\gamma\omega$ must be zero if $a_i\neq a$. Since the periods of ω cannot be all equal to zero, there must exist some a_i equal to a such that the corresponding eigen-period of ω is nonzero, and hence we can use it to projectivize the domain of periods as well as the induced action. Then each of the exponents of the projectivized action is $a_i'=a_i-a \bmod k$, where we use representatives with $0\le a_i'<k$ throughout.

The first lemma below gives a lower bound for the age contribution of a τ-orbit of zeros or poles in the context of Proposition 4.1.

Lemma 4.5. *Let τ be a non-trivial automorphism of order k of a (stable) pointed smooth curve (X,Z,P), fixing zeros and poles setwise (but not necessarily pointwise) and fixing projectively an Abelian differential ω with the zeros Z and poles P. Let $\{x_1,\dots,x_{k'}\}$ be a τ-orbit of unlabeled zeros or poles, where $k=k'\ell$. Consider the subspace $U\subseteq\mathbb{A}_{\mathrm{rel}}(X)$ generated by the $k'-1$ relative periods joining the x_i if they are zeros, or by the $k'-1$ loops at each of the x_i if they are poles. Then for $a\in\{0,\dots,k-1\}$ as in (17) we have*

$$\operatorname{age}(\tau|_U) \;\ge\; \frac{1}{k}\sum_{i=1}^{k'-1}(\ell i-a)\bmod k \;\ge\; \frac{1}{2k'}(k'-2)(k'-1)\,. \tag{I}$$

In particular, in this case $\operatorname{age}(\tau)\ge 1$ *if* $k'\ge 5$.

Proof. The unprojectivized eigenvalues of the τ-action restricted to the subspace U are $\zeta_{k'}, \dots, \zeta_{k'}^{k'-1}$, where $\zeta_{k'}$ is a primitive k'-th root of unity. In other words, these eigenvalues are $\zeta_k^{\ell}, \dots, \zeta_k^{\ell(k'-1)}$, which thus implies the first inequality. Note that the sum in the bound consists of $k'-1$ distinct numbers in $[0, k-1]$ that belong to the same congruence class mod ℓ. Thus its minimum is attained at $\frac{1}{k'}\sum_{j=0}^{k'-2} j$. $\square$

In the next lemma we are in the context of an automorphism $\boldsymbol{\tau}$ of a multi-scale differential compatible with Γ, where Γ is a level graph corresponding to a boundary component of $\overline{B}$. Consider a vertex v of the graph and denote by (X, ω, Z, P) the differential associated to it. We define the *multi-vertex RC-independent subspace* $V \subset H_1(X \setminus P, Z; \mathbb{C}) \oplus \mathbb{C}_{\mathrm{hor}}(X)$ to be the largest subspace such that periods of (X, ω) in V (including residues of the poles and plumbing parameters for horizontal edges of X) can vary independently without being constrained by the other vertices of Γ due to global and matching residue conditions. In other words, V is the largest subspace of parameters associated to X such that the dimension of $\bigoplus_{i=0}^{d-1} \tau^i(V)$ does not drop after imposing the residue conditions $\mathfrak{R}$ and matching horizontal plumbing parameters (in case a horizontal edge joins two permuted vertices). Note that V always contains the absolute periods of X and relative periods that join between zeros of X. We denote by M the dimension of V.

Lemma 4.6. *Suppose $\boldsymbol{\tau}$ permutes d vertices of Γ and let τ be its restriction to the homology of these vertices. Let M be the dimension of the multi-vertex RC-independent subspace of each vertex. Then*

$$\mathrm{age}(\tau) \;\geq\; \frac{M}{2}(d-1)\,.$$

In particular, $\mathrm{age}(\tau) \geq 1$ *if $d \geq 3$, $M > 0$ and if $d = 2$, $M > 1$.*

If a permuted vertex has genus g with n zero edges, then $M \geq 2g+n-1$ because V contains the subspace of absolute and relative periods. In particular, the case $M \leq 1$ can only occur for $g = 0$ with $n \leq 2$.

Proof of Lemma 4.6. The automorphism $\boldsymbol{\tau}$ permutes these d vertices and for each one we have an M-dimensional subspace V of homology which is cyclically permuted among the vertices. Then the restricted action of τ to the sum of these M-dimensional subspaces can be described in a suitable basis as

$$\begin{pmatrix} 0_{M\times M(d-1)} & A_{M\times M} \\ I_{M(d-1)\times M(d-1)} & 0_{M(d-1)\times M} \end{pmatrix} \tag{18}$$

where A is the matrix representing the automorphism τ^d acting on V. Suppose the action of τ^d on V has order k. If the eigenvalues of A are given by $\zeta_k^{a_i}$ for $i = 1, \dots, M$, then the eigenvalues of the full matrix are given by $\zeta_{dk}^{a_i+jk}$ for all $i = 1, \dots, M$ and $j = 0, \dots, d-1$. In the case of $M > 0$, there exists a nonzero eigen-period corresponding to certain eigenvalue $\zeta_{dk}^{a_{i_0}+jk}$, which can be used to projectivize the action. Therefore, $\mathrm{age}(\tau)$ is bounded below by M sums of type $\frac{1}{dk}\sum_{j=0}^{d-1}(jk + a') \bmod dk$ for some $a' = a_i - a_{i_0}$. Since each sum consists of the representatives of the same congruence class ($a' \bmod k$) in consecutive subintervals of length k in $[0, dk-1]$, its minimum is attained at $\frac{1}{dk}\sum_{j=0}^{d-1} jk = (d-1)/2$. We thus conclude that $\mathrm{age}(\tau) \geq M(d-1)/2$. $\square$

The following third lemma gives a lower bound for the age contribution of a $\boldsymbol{\tau}$-orbit of zeros or poles from the permutation representation on the space of residues and relative cycles on a $\boldsymbol{\tau}$-fixed vertex and also on the rest of a multi-scale differential whose marked points are fixed. We define $\mathbb{A}_{\rm rel}(X_{>-J}) = \prod_{-i>-J}\mathbb{A}_{\rm rel}(X_{-i})$ and similarly for $X_{<-J}$ to parameterize periods of the vertices above or below level $-J$ (up to level-wise projectivization).

Lemma 4.7. *Let Γ be a level graph representing a boundary stratum in $\overline{B}$ and let $\boldsymbol{\tau} = (\tau_{(-i)})$ be an automorphism of a projectivized multi-scale differential compatible with Γ. Suppose $\boldsymbol{\tau}$ fixes a vertex v at level $-J$ of Γ. Denote the restriction of the multi-scale differential to v by (X,ω,Z,P) and suppose that the order of $\tau = \boldsymbol{\tau}|_v$ is $k = k'\ell$. Then the following estimates hold:*

(i) *Let $\{x_1,\ldots,x_{k'}\}$ be a $\boldsymbol{\tau}$-orbit of unlabeled non-simple poles of (X,ω) corresponding to the lower ends of edges ending at v. Consider the subspace $U \subseteq \mathbb{A}_{\rm rel}(X_{(-J)})$ generated by the loops around these poles constrained by the residue conditions $\mathfrak{R}$ imposed by the higher levels of Γ. Suppose that these edges are adjacent to d connected components of the graph $\Gamma_{>-J}$ at higher level. Then*

$$\operatorname{age}(\boldsymbol{\tau}|_U) \;\geq\; \frac{1}{k}\sum_{\substack{i=1\\(k'/d)\nmid i}}^{k'-1} (\ell i - a) \bmod k\,, \tag{P1}$$

$$\operatorname{age}(\boldsymbol{\tau}|_{\mathbb{A}_{\rm rel}(X_{(>-J)})}) \;\geq\; d-1\,. \tag{P2}$$

(ii) *Let $\{x_1,\ldots,x_{k'}\}$ be a $\boldsymbol{\tau}$-orbit of unlabeled zeros of (X,ω) corresponding to higher ends of edges adjacent to v. Let $U \subseteq \mathbb{A}_{\rm rel}(X_{(-J)})$ be the subspace generated by the relative periods between these zeros. Then*

$$\operatorname{age}(\boldsymbol{\tau}|_U) \;\geq\; \frac{1}{k}\sum_{i=1}^{k'-1}(\ell i - a) \bmod k \;\geq\; \frac{1}{2k'}(k'-2)(k'-1)\,. \tag{Z}$$

(iii) *Let $\{x_1,\ldots,x_{k'}\}$ be a $\boldsymbol{\tau}$-orbit of unlabeled simple poles of (X,ω) corresponding to horizontal edges, each of which has exactly one end adjacent to v at the x_i. Let $U \subseteq \mathbb{A}_{\rm rel}(X_{(-J)})$ be the subspace generated by the residue cycles at these poles. Then*

$$\operatorname{age}(\boldsymbol{\tau}|_{\mathbb{C}_{\rm hor}}) \;\geq\; \frac{1}{k'}\sum_{j=0}^{k'-1} j \bmod k' \;=\; \frac{k'-1}{2}\,, \tag{SPH1}$$

$$\operatorname{age}(\boldsymbol{\tau}|_U) \;\geq\; \frac{1}{k'}\sum_{j=1}^{k'-1}(j-a) \bmod k' \;\geq\; \frac{1}{2k'}(k'-2)(k'-1)\,. \tag{SPH2}$$

(iv) *Let $\{x_1,\ldots,x_m,y_1,\ldots,y_m\}$ be a $\boldsymbol{\tau}$-orbit (i.e. $k'=2m$) of unlabeled simple poles of (X,ω) corresponding to horizontal edges both of whose ends x_i and y_i are adjacent to v (i.e. they form self-nodes as m loops in the dual graph at v). Let $U \subseteq \mathbb{A}_{\rm rel}(X_{(-J)})$ be the subspace generated by the residue cycles at these poles. Then*

$$\operatorname{age}(\boldsymbol{\tau}|_{\mathbb{C}_{\rm hor}}) \;\geq\; \frac{1}{k'}\sum_{j=0}^{m-1} 2j \bmod k' = \frac{m-1}{2}\,, \tag{SPS1}$$

$$\operatorname{age}(\boldsymbol{\tau}|_U) \;\geq\; \frac{1}{k'} \sum_{j=1}^{m-1} (2j-a) \bmod k' \geq \frac{1}{2m}(m-1)^2 \,. \tag{SPS2}$$

In all of the above a is analogously defined as in (17) and in the last inequality it is an odd number.

Proof. In the setting of (i), the poles x_i can be grouped accordingly into d sets each of which has k'/d elements adjacent to the same connected component of $\Gamma_{>-J}$. Let γ_i be a loop around each x_i and consider the space generated by all the γ_i. Applying the GRC, the subspace U is cut out by the equations $\sum_{j=1}^{k'/d} \gamma_{i+jd} = 0$, for $i = 1, \ldots, d$. Then $(\tau_{(-J)})|_U$ is the regular representation of the cyclic group of order k' with all representations induced from the subgroup of order k'/d removed. This means that the eigenvalues of $(\tau_{(-J)})|_U$ are $\zeta_{k'}^i = \zeta_k^{\ell i}$, for all $i \neq jk'/d$ with $j = 1, \ldots, d$. Hence $\operatorname{age}(\tau_{(-J)})|_U \geq \frac{1}{k} \sum_{i \in \{1,\ldots,k'\} : (k'/d) \nmid i} (\ell i - a) \bmod k$ for some $a \in \{0, \ldots, k-1\}$ where ζ_k^a is the eigenvalue for the eigenform ω under the action of $\tau_{(-J)}$. This shows (P1).

For the age estimate on higher level, the automorphism $\boldsymbol{\tau}$ permutes the top vertices of the d connected components of $\Gamma_{>-J}$. Hence there exist d' disjoint (locally) top vertices on some level $-J' > J$ that are permuted by $\tau_{(-J')}$ where $d \operatorname{mid} d'$. Note that (locally) top vertices do not admit any vertical polar edges, hence they are not constrained by the GRC. If the multi-vertex RC-independent subspace of these d' vertices has dimension $M \geq 2$, then (P2) follows from Lemma 4.6. The only possibility for $M \leq 1$ is when these d' vertices have genus zero and each admits some horizontal edges joining between them (so that we need to match horizontal residues and plumbing parameters between them). Since these d' vertices belong to d connected components, in this case $d' \geq 2d$ and we obtain at least d independent residue cycles and d independent plumbing parameters (from at least $d'/2 \geq d$ horizontal edges) that are permuted by $\tau_{(-J')}$, which yields the desired bound by applying the argument of Lemma 4.6 for the case $M = 2$.

In case (ii) of zeros the argument for (Z) is the same as for (P1), using relative periods joining the zeros instead of loops around the poles. Since there are no residue conditions, only the trivial representation ($i = 0$) has to be omitted.

In case (iii) of horizontal nodes, suppose each horizontal edge has local equation $x_i y_i = h_i$ for $i = 1, \ldots, k'$, where h_i is the plumbing parameter, such that $x_1 \mapsto x_2, x_2 \mapsto x_3, \ldots, x_{k'} \mapsto \zeta_\ell^x x_1$ under $\boldsymbol{\tau}$, where the last one is due to that the action of $\boldsymbol{\tau}^{k'}$ multiplies x_1 by an ℓ-th root of unity. Similarly suppose $y_1 \mapsto y_2, y_2 \mapsto y_3, \ldots, y_{k'} \mapsto \zeta_\ell^y y_1$ under $\boldsymbol{\tau}$. It follows that $h_1 \mapsto h_2, h_2 \mapsto h_3, \ldots, h_{k'} \mapsto \zeta_\ell^b h_1$ under $\boldsymbol{\tau}$ for some $b = x + y \in \{0, \ldots, \ell - 1\}$. Then the associated eigenvalues as k-th roots of unity have exponents $b, b + \ell, \ldots, b + (k'-1)\ell \bmod k$ for $k = k'\ell$. Since in this case $\boldsymbol{\tau}^{k'}$ fixes a simple pole branch, it implies that $(\boldsymbol{\tau}^{k'})^*\omega = \omega$ restricted to v, and hence a is divisible by ℓ. Altogether it gives

$$\operatorname{age}(\boldsymbol{\tau}|_{\mathbb{C}_{\mathrm{hor}}}) \;\geq\; \frac{1}{k} \sum_{j=0}^{k'-1} (b + j\ell) \bmod k \,.$$

Clearly $b = 0$ minimizes the bound, which gives (SPH1). For (SPH2), the proof is similar. The only possible difference is that the sum of the residue cycles might be trivial due to the Residue Theorem (combined with GRC from higher level if v

has other poles). Hence we can only obtain a bound by summing up $k'-1$ terms (instead of k'), which minimizes at $\frac{1}{k'}\sum_{j=0}^{k'-2} j$ as seen before.

In case (iv) we can assume that $\boldsymbol{\tau}(x_i) = x_{i+1}$, $\boldsymbol{\tau}(y_i) = y_{i+1}$, $\boldsymbol{\tau}(x_m) = y_1$ and $\boldsymbol{\tau}(y_m) = x_1$ by labeling the points appropriately. In particular $\boldsymbol{\tau}^m$ swaps x_i and y_i for all i. The opposite residue condition implies that $(\boldsymbol{\tau}^m)^*\omega = -\omega$. Consequently a is odd. Using similar arguments as in (iii), we thus obtain (SPS1) and (SPS2) respectively from the cyclic group action on the m horizontal plumbing parameters and from the residue cycles at the self-nodes formed by gluing each x_i with y_i. $\square$

Finally we need the following estimate for sums analogous to those appearing in the preceding lemmas.

Lemma 4.8. *Let $\phi_k(a) = \sum_n (n-a) \bmod k$ where the sum ranges over all the $\varphi(k)$ integers $1 \le n < k$ that are relatively prime to k. Then $\phi_k(a) \ge k$ for any a as long as $k \ne 2, 3, 4, 6$.*

Proof. The set of integers relatively prime to k equidistributes in the intervals $(0, a]$ and $(a, k]$, in fact in any interval, with an effective error rate of the number $d(k)$ of divisors of k (see [BIR08, Lemma 1.4] and simplify the argument therein by removing the extra congruence condition). Consequently, $\phi_k(a) \to k\varphi(k)/2$ for any a as $k \to \infty$ with controlled error terms. It thus suffices to check small values of k, which gives the list above. $\square$

We can now prove Proposition 4.1 and Proposition 4.2 at the same time. The main difference to keep in mind is that we can use the inequality (I) of Lemma 4.5 in the context of Proposition 4.1, while we have to use the inequalities of Lemma 4.7 in the context of Proposition 4.2 due to possible residue conditions. We will also skip the verification of realizability and congruence condition of each case in the following already lengthy proof (see Remark 4.4 if the reader is interested).

Throughout the proof we stick to the following notations. Let X' be the quotient of X by the group action generated by an automorphism τ of order k, and let g' be the genus of X'. Then the associated map $\pi\colon X \to X'$ is a cyclic cover of degree k with the deck transformation group generated by τ. Let Z' and P' be the π-images of Z and P respectively. Let b be the number of branch points and s_i, for $i = 1, \ldots, b$, be the cardinality of each ramified fiber. Note that every s_i divides k since π is a cyclic cover. Moreover in this case the Riemann–Hurwitz formula gives

$$2g - 2 \;=\; k(2g' - 2) + bk - \sum_{i=1}^{b} s_i \,. \tag{19}$$

We say that a fiber of π is *special* if it consists of zeros or poles of ω. Moreover, a special fiber is called a zero (resp. pole) fiber if it consists of zeros (resp. poles) of ω. Note that a special fiber does not have to consist of ramification points, and conversely, a ramified fiber does not have to be special.

Proof of Proposition 4.1 and Proposition 4.2. Let Γ be a level graph representing a boundary stratum in $\overline{B}$ and let $\boldsymbol{\tau} = (\tau_{(-i)})$ be an automorphism of a projectivized multi-scale differential compatible with Γ fixing a vertex v. We denote by τ the restriction of $\boldsymbol{\tau}$ to v, and by (X, ω, Z, P) the restriction of the multi-scale differential to v. Let $\zeta^{a_1}, \ldots, \zeta^{a_N}$ be the eigenvalues for the induced action of τ on the homology $H_1(X \setminus P, Z; \mathbb{C})^{\mathfrak{R}}$ (and on thus also the cohomology $H^1(X \setminus P, Z; \mathbb{C})^{\mathfrak{R}}$), where $\mathfrak{R}$ are the residue conditions in the general situation of Proposition 4.2.

Recall that if $\tau^*\omega = \zeta^a\omega$, then we can use a nonzero eigen-period corresponding to the eigenvalue ζ^a to projectivize the induced action (restricted to the level of v). Then each of the exponents of the projectivized action is $a_i' = a_i - a \bmod k$, where $0 \le a_i' < k$.

The τ-invariant subspace of $H^1(X \setminus P, Z; \mathbb{C})^{\mathfrak{R}}$ can be identified with a subspace of the cohomology of the quotient surface $H^1(X' \setminus P', Z'; \mathbb{C})$, cut out by some residue constraints $\mathfrak{R}'$. Independently of $\mathfrak{R}'$, the action of τ preserves the symplectic paring of the absolute homology $H_1(X; \mathbb{C})$. Hence the $2g$ eigenvalues from the absolute part split into g conjugate pairs of type $(\zeta^{a_i}, \zeta^{k-a_i})$ for $0 < a_i \le [k/2]$ and pairs of type $(1,1)$ if $a_i = 0$. In particular, the sum of the exponents from the absolute part is divisible by k, and is at least k unless all absolute periods are τ-invariant.

***Case* $\mathbf{g' \neq 0}$*,* $\mathbf{a = 0}$.** First consider the case $a = 0$, i.e. ω is a τ-invariant form. If $g > g'$, then $H^1(X'; \mathbb{C}) \to H^1(X; \mathbb{C})$ is not onto for the dimension reason, hence the absolute periods are not all invariant. By the preceding paragraph, we conclude that $\operatorname{age}(\tau) \ge k/k = 1$.

The opposite case $g \le g'$ is only possible by Riemann–Hurwitz if $g = g' = 1$ or $g = g' = 0$. Suppose $g = g' = 1$. Then π is an elliptic isogeny with no ramification, and consequently there is at least one unramified zero fiber. Applying to this fiber (I) or (Z) (for $a = 0$, $\ell = 1$ and $k' = k$ therein), we obtain that $\operatorname{age}(\tau) \ge 1$ if $k \ge 3$. For $k = 2$, we also get $\operatorname{age}(\tau) \ge 1$ if there are at least two special unramified zero fibers by the same reason, or if the second special unramified fiber consists of poles (with possible residue constraints) by (P1), (P2) or (SPH1). For $k = 2$ and only one special fiber, the map π is a bielliptic cover and the corresponding stratum is $\mathbb{P}\Omega\mathcal{M}_1(\{0,0\})$, where the two zeros z_1 and z_2 (of order zero) are exchanged by τ. In this case τ induces a quasi-reflection, listed as **Case (1)**.

The other case is $g = g' = 0$, and we leave it to be discussed later.

***Case* $\mathbf{g' \neq 0}$*,* $\mathbf{0 < a < k}$.** Next consider the case $0 < a < k$. If $a \le k/2$, from the subspace of invariant periods we obtain $\operatorname{age}(\tau) \ge \frac{1}{2} \dim H^1(X' \setminus P', Z'; \mathbb{C})^{\mathfrak{R}'}$ which is at least one except for $g' = 0$, and we again leave this exceptional case to be discussed later.

If $a > k/2$, consider any conjugate pair of eigenvalues with exponents a_i and $k - a_i$ from the absolute part, where $a_i \le k/2$ (or the pair $(1,1)$ with ζ-exponent 0). Then after subtracting a and normalizing to the range $[0, k-1]$, this pair contributes at least $2/k$ to $\operatorname{age}(\tau)$. Hence the g pairs contribute at least $2g/k$. By Riemann–Hurwitz, $2g - 2 \ge (2g' - 2)k$, hence $\operatorname{age}(\tau) \ge 2g/k \ge 1$ if $g' > 1$.

It remains to discuss $g' = 1$. In this case $2g - 2 = (k - s_1) + \cdots + (k - s_b)$ by Riemann–Hurwitz (19), where recall that b is the number of branch points of π, s_i is the cardinality of each ramified fiber, and every s_i divides k. If $b \ge 2$, then $2g - 2 \ge k/2 + k/2 = k$ and consequently $\operatorname{age}(\tau) \ge 2g/k > 1$.

Suppose $b = 1$ and we set $s = s_1$. Then $2g - 2 = k - s \ge k/2$ and consequently $g \ge 2$. If the unique ramified fiber is not a special zero fiber, then there must exist a special unramified zero fiber. The contribution (I) or (Z) (with $k = k'$) from this unramified zero fiber gives $\operatorname{age}(\tau) \ge 1$ for $k \ge 4$ in the case of $a > k/2$. For $k \le 3$, the estimate $2g/k \ge 4/3 > 1$ justifies $\operatorname{age}(\tau) > 1$ in this case.

Now suppose that the ramified fiber is a special zero fiber. Applying (I) or (Z) to this fiber (with $s = k'$), if $s \ge 3$, then it contributes at least $(s-2)/s \ge (s-2)/k$, and hence $\mathrm{age}(\tau) \geq (2g+s-2)/k = 1$ in this case. If $s \leq 2$, then $2g \geq k$ and from the absolute periods we obtain $\mathrm{age}(\tau) \geq 2g/k \geq 1$ in this case.

***Case* $\mathbf{g}' = \mathbf{0}$.** Finally consider the case $g' = 0$, i.e. π is a cyclic cover of $\mathbb{P}^1$ of degree k with b branch points. We discuss various cases according to the number of the branch points b. For convenience we denote by S_{ram} and S_{un} the sets of ramified and unramified special fibers respectively.

***Case* $\mathbf{g}' = \mathbf{0},\ \mathbf{k} \geq \mathbf{3},\ \mathbf{b} = \mathbf{2}$.** Suppose $b = 2$. Then $X \cong \mathbb{P}^1$ and π is totally ramified at two points. Since the stability of X in a stratum of genus zero implies $|Z \cup P| \geq 3$, there is at least one special unramified fiber, i.e. $|S_{\mathrm{un}}| \geq 1$. If an unramified special fiber consists of zeros, the age contribution from (I) or (Z) implies that $\mathrm{age}(\tau) \geq 1$ for $k \geq 5$. Moreover, if an unramified special fiber consists of simple poles, then using (SPH1) or (SPS1) we obtain $\mathrm{age}(\boldsymbol{\tau}) \geq 1$ for $k \geq 3$. If an unramified special fiber consists of higher order poles, recall that for such a fiber d is the number of connected components of $\Gamma_{>-J}$ adjacent to these k poles as in Lemma 4.7. If $d \geq 3$, then by (P2) we obtain $\mathrm{age}(\boldsymbol{\tau}) \geq 1$. If $d \leq 2$, by combining (P1) and (P2) we obtain $\mathrm{age}(\boldsymbol{\tau}) \geq 1$ for $k \geq 5$. The remaining cases are thus $k \leq 4$.

Subcase $\mathbf{k} = \mathbf{3}$. By the estimates in Lemma 4.5 and Lemma 4.7, any unramified special fiber in all situations (with or without residue constraints) contributes at least $1/3$ to the age. Therefore, if $|S_{\mathrm{un}}| \geq 3$, we thus obtain that $\mathrm{age}(\boldsymbol{\tau}) \geq 1$. Moreover, the case $|S_{\mathrm{ram}}| = 0$ is impossible since the sum of entries of μ (which is -2) is not divisible by 3. Hence we only need to consider the cases $|S_{\mathrm{un}}| = 1$ or 2 and $|S_{\mathrm{ram}}| = 1$ or 2.

If $|S_{\mathrm{un}}| = 2$ and $|S_{\mathrm{ram}}| = 1$ or 2, then by the dimension reason there is an extra subspace of periods (besides the two unramified fiber contributions) on which τ acts with eigenvalues 1 or $(1,1)$ or (ζ, ζ^2) since the total determinant of the τ-action is one. One checks that $\mathrm{age}(\tau) \geq 1$ in all these cases.

Consider now the case $|S_{\mathrm{un}}| = 1$ and $|S_{\mathrm{ram}}| = 2$. Assume there is a totally ramified zero. If the second totally ramified fiber is a (GRC free) pole or a zero, then besides the eigenvalues ζ, ζ^2 from the unramified fiber we get an eigenvalue 1 from the residue or a relative period, since the determinant of the unprojectivized action of τ is one. This gives $\mathrm{age}(\tau) \geq 1$. If the second ramified fiber is a (GRC constrained) pole with $d = 1$, we get $\Omega\mathcal{M}_0^{\mathfrak{R}}(3m_1+m_2-2, -m_2, \{-m_1, -m_1, -m_1\})$, with $m_i > 0$ and $\mathfrak{R} = \{r_3 + r_4 + r_5 = 0,\ r_2 = 0\}$, which is **Case (R1)**.

Suppose both ramified fibers are poles. Then the unramified special fiber must consist of zeros, hence it gives signatures of type $(\{m, m, m\}, -m_1, -m_2)$ with $3m - m_1 - m_2 = -2$ and $m_1, m_2 > 0$, where the unramified zero fiber contributes eigenvalues ζ and ζ^2. If any of the two poles can have a nonzero residue, then the residue gives an extra eigenvalue 1, which together with ζ and ζ^2 makes $\mathrm{age}(\tau) \geq 1$. The remaining case leads to the GRC-constrained stratum with $\mathfrak{R} = \{r_4 = 0\}$ or $\mathfrak{R} = \{r_4 = r_5 = 0\}$ listed as **Case (R2)**.

Finally if $|S_{\mathrm{un}}| = 1$ and $|S_{\mathrm{ram}}| = 1$, it gives $\Omega\mathcal{M}_0(\{m, m, m\}, -3m-2)$ with $m \in \mathbb{Z}$, which is **Case (2)**.

Subcase $\mathbf{k} = \mathbf{4}$. Suppose first an unramified fiber consists of poles that are subject to residue conditions. As noticed above, these are vertical edges joined to d higher connected components with $d > 1$. Then combining inequalities (P1) and (P2) gives $\mathrm{age}(\boldsymbol{\tau}) \geq 1$ in this case.

Now we can assume that any unramified special fiber consists of either zeros or poles without extra residue constraints (i.e. $d = 1$). Using (I), (Z), or (P1), the relative periods or residue cycles from this fiber contribute at least $3/4$ to $\mathrm{age}(\tau)$. Hence we can further assume that $|S_{\mathrm{un}}| = 1$. If $|S_{\mathrm{ram}}| = 2$, then X has genus $g = 0$ with six zeros and poles, and hence the (unprojectivized) stratum of X has dimension equal to four. Besides the eigenvalues ζ, ζ^2, ζ^3 from the special unramified fiber, the remaining eigenvalue must be 1 or $\zeta^2 = -1$, since the determinant of the τ-action is ± 1 (from an invertible integral matrix). In both cases one checks that $\mathrm{age}(\tau) \geq 1$. Therefore, the remaining possibility is $|S_{\mathrm{un}}| = |S_{\mathrm{ram}}| = 1$, i.e. $\mathbb{P}\Omega\mathcal{M}_0(\{m,m,m,m\}, -4m-2)$ where the four zeros (or poles) are permuted by τ and the pole (or zero) is fixed, with (unprojectivized) eigenvalues ζ, ζ^2, ζ^3 and (projectivized) $\mathrm{age}(\tau) = 3/4 < 1$ if $a = 1$. This is **Case (8)**.

Case $\mathbf{g' = 0}$, $\mathbf{k \geq 3}$, $\mathbf{b \geq 3}$. It is well-known (e.g. [DM86, Proposition 2.3.1]) that any primitive k-th root of unity appears as an eigenvalue with multiplicity $b-2 \geq 1$ for the action of τ on the $2g$-dimensional absolute part $H^1(X;\mathbb{C})$, and hence $g \geq 1$.
Subcase $\mathbf{k \notin \{2,3,4,6\}}$. For $k \notin \{2,3,4,6\}$, using Lemma 4.8 and its notation we find that $\mathrm{age}(\tau) \geq (b-2)\phi_k(a)/k \geq 1$. Hence the remaining cases are cyclic covers of $\mathbb{P}^1$ with degree $k = 2,3,4,6$, with $b \geq 3$ branch points, and $g \geq 1$.
Subcase $\mathbf{k = 3}$. In this case the cover is totally ramified at $b = g+2$ points. If moreover $g \geq 3$, since $\phi_3(a) \geq 1$ for any a, we find $\mathrm{age}(\tau) \geq (b-2)/3 \geq 1$. Hence we only need to consider $g = 2$ and $g = 1$.

Consider first $g = 2$. Then the absolute periods already give a contribution of at least $2/3$ to $\mathrm{age}(\tau)$. If $|S_{\mathrm{un}}| \geq 1$, then an unramified special fiber contributes to the age at least $1/3$ by the estimates in Lemma 4.5 and Lemma 4.7, and hence $\mathrm{age}(\boldsymbol{\tau}) \geq 1$ in this case. We can thus assume that $|S_{\mathrm{un}}| = 0$. Moreover, if there are two ramified zero fibers, then the relative paths joining them give an eigenvalue 1. If there is any ramified pole fiber whose residue is not constrained to be zero, then its residue gives an eigenvalue 1. In both cases $\mathrm{age}(\tau) \geq 1$ since $1, \zeta, \zeta^2$ all appear as eigenvalues. The remaining cases are $|S_{\mathrm{ram}}| = 2,3,4$ with exactly one ramified zero fiber and all residues constrained to be zero, giving **Cases (3)**, **(R3)** and **(R4)**.

Now we deal with $g = 1$. The absolute periods give eigenvalues ζ, ζ^2, and hence contribute at least $1/3$ to $\mathrm{age}(\tau)$. Note that any two special zero fibers contribute an eigenvalue 1 using their relative periods, which makes the total age at least one together with the absolute periods. Similarly if there is a totally ramified pole fiber whose residue is not constrained to be zero, then its residue gives an eigenvalue 1, which again makes the total age at least one. We can thus assume that there is exactly one special zero fiber and all ramified poles are constrained to have zero residue. Moreover if $|S_{\mathrm{un}}| \geq 2$, since each unramified fiber contributes at least $1/3$ to the age by the estimates in Lemma 4.5 and Lemma 4.7, then altogether we obtain $\mathrm{age}(\boldsymbol{\tau}) \geq 1$. Hence we only need to consider $|S_{\mathrm{un}}| = 1$ or 0.

Suppose $|S_{\mathrm{un}}| = 1$, which contributes at least $1/3$ to the age as explained above. If this is an unramified pole fiber giving three edges that permute three connected components in upper level, then (P2) contributes 1 to the age, hence we can assume that the unramified fiber is either a zero fiber or a pole fiber not constrained by the GRC. The remaining cases give signatures $(\{0,0,0\})$ or $(\{m,m,m\}, -3m)$ without residue conditions and those with residue conditions as follows:

- $(\{m,m,m\}, -m_1, -m_2)^{\mathfrak{R}}$ with $\mathfrak{R} = \{r_4 = 0\}$ or $\mathfrak{R} = \{r_4 = r_5 = 0\}$,

- $(\{m,m,m\},-m_1,-m_2,-m_3)^{\mathfrak{R}}$ with residue condition $\mathfrak{R}=\{r_4=r_5=0\}$ or $\mathfrak{R}=\{r_4=r_5=r_6=0\}$,
- $(3m_1+m_2,\{-m_1,-m_1,-m_1\},-m_2)^{\mathfrak{R}}$ with $\mathfrak{R}=\{r_5=0\}$,
- $(3m_1+m_2+m_3,\{-m_1,-m_1,-m_1\},-m_2,-m_3)^{\mathfrak{R}}$ with $\mathfrak{R}=\{r_5=r_6=0\}$.

The first two cases without residue conditions correspond to **Cases (6)** and **(7)**. However in the cases with non-trivial $\mathfrak{R}$, the vertex v already contributes at least $2/3$ to $\mathrm{age}(\boldsymbol{\tau})$. If the three edges in the unramified special fiber join three vertices in Γ that are permuted by $\boldsymbol{\tau}$, then $\mathrm{age}(\boldsymbol{\tau})\geq 1$ by the argument in the proof of Proposition 4.11 below. If they join the same vertex v', then their relative periods or residues (without GRC) for v' contribute at least $1/3$ to $\mathrm{age}(\boldsymbol{\tau})$, thus making $\mathrm{age}(\boldsymbol{\tau})\geq 1$. Hence these cases do not appear in the tables.

Suppose $|S_{\mathrm{un}}|=0$, i.e. there is no unramified special fiber. If there is no pole, then the special fibers consist of a unique zero fiber, thus giving the signature $\mu=(0)$ listed as **Case (4)**. If there are (ramified) poles, then each of them is constrained to have residue zero as explained above, hence we obtain signatures $(m,-m)$ and $(m_1+m_2,-m_1,-m_2)^{\mathfrak{R}}$ with $\mathfrak{R}=\{r_2=0\}$ or $\{r_2=r_3=0\}$, which correspond to **Cases (5)** and **(R5)**.

Subcase $\mathbf{k=4}$. We have $\phi_4(a)\geq 2$ for any a, and hence $\mathrm{age}(\tau)\geq 2(b-2)/4\geq 1$ for $b\geq 4$. We thus need to consider the case $b=3$. Since $2g+6=\sum_{i=1}^{3}(4-s_i)\leq 9$ with s_i a proper divisor of 4, we conclude that $g\leq 1$. Since there exist eigenvalues ζ and ζ^3 from the absolute homology, we only need to consider the case $g=1$.

In this case π has two totally ramified fibers and the third ramified fiber consists of two simply ramified points. Since the absolute periods already contribute eigenvalues ζ and ζ^3 (thus at least $1/2$ to the age), if there exists an unramified special fiber, it contributes by Lemma 4.7 enough to make $\mathrm{age}(\boldsymbol{\tau})\geq 1$. If there are two special zero fibers, then their relative periods give an eigenvalue 1, which contributes enough to make $\mathrm{age}(\boldsymbol{\tau})\geq 1$. Similarly if there is a totally ramified pole without residue constraint, then its residue gives an eigenvalue 1, which also makes $\mathrm{age}(\boldsymbol{\tau})\geq 1$. Hence we can assume in the sequel that $|S_{\mathrm{un}}|=0$, there is a unique special zero fiber, and any totally ramified pole is constrained to have residue zero.

If the fiber with two simply ramified points is not special, we get the strata $\mathbb{P}\Omega\mathcal{M}_1(0)$ and $\mathbb{P}\Omega\mathcal{M}_1(m,-m)$ which are listed as **Cases (9)** and **(10)**.

If the fiber consisting of two simply ramified points is a special zero fiber, then it contributes at least an eigenvalue ζ^2. As said, any remaining special fiber must be a ramified pole fiber with residue constraint. We thus get $(\{0,0\})$, $(\{m,m\},-2m)$ and $(\{m,m\},-m_1,-m_2)^{\mathfrak{R}}$ with $\mathfrak{R}=\{r_3=0\}$ or $\{r_3=r_4=0\}$, listed as **Cases (11)**, **(12)** (with two permuted zeros) and **(R6)**.

Now suppose the fiber with two simply ramified points is a special pole fiber. A GRC with the case $d=2$ for this fiber (as defined in Lemma 4.7 (i)) gives enough to make $\mathrm{age}(\boldsymbol{\tau})\geq 1$ by (P2). The case $d=1$ leads to the strata $\mathbb{P}\Omega\mathcal{M}_1(\{-m,-m\},2m)$ and $\mathbb{P}\Omega\mathcal{M}_1^{\mathfrak{R}}(2m_1+m_2,\{-m_1,-m_1\},-m_2)$ with $\mathfrak{R}=\{r_4=0\}$ or $\mathfrak{R}=\{r_2+r_3=0\}$, listed as **Cases (12)** (with two permuted poles) and **(R7)**.

If the fiber with two simply ramified points consists of simple poles adjacent to other vertices, then the residues at the two nodes give eigenvalues ± 1. Together with the eigenvalues ζ,ζ^3 from the absolute periods we obtain $\mathrm{age}(\tau)\geq 1$ for any ζ^a used for projectivization. If the fiber with two simply ramified points has two simple poles that form a self-node, then ω is τ-anti-invariant, i.e. $a=2$. Projectivization of the eigenvalues of the absolute periods already gives $\mathrm{age}(\tau)\geq 1$.

Subcase $\mathbf{k = 6}$. In this case $\phi_6(a) \geq 4$ for any $a \neq 5$ and $\phi_6(5) = 2$, hence $\mathrm{age}(\tau) \geq 2(b-2)/6 \geq 1$ for $b \geq 5$ and any a. Moreover, $\mathrm{age}(\tau) \geq 1$ for $b = 4$ and any $a \neq 5$.

Consider first $b = 4$ and $a = 5$. Then the eigenvalues of τ from the absolute homology of X contain $\zeta, \zeta, \zeta^5, \zeta^5$, contributing $2/3$ to $\mathrm{age}(\tau)$, and $g \geq 2$ in this case. If $g > 2$, an additional conjugate pair of eigenvalues from the absolute part of type $(\zeta^{a_i}, \zeta^{6-a_i})$ for $1 \leq a_i \leq 3$ or $(1, 1)$, after dividing by ζ^5, can contribute at least $1/3$ to $\mathrm{age}(\tau)$, which is enough. For $g = 2$, if there is a special zero or pole fiber with cardinality 2 or 3 or 6, using the respective estimates in Lemma 4.5 and Lemma 4.7, we can still obtain that $\mathrm{age}(\boldsymbol{\tau}) \geq 1$. If all zeros and poles are totally ramified, then by Riemann–Hurwitz the only case is $\mathbb{P}\Omega\mathcal{M}_2(2)$ for $s_1 = 1$ and $s_2 = s_3 = s_4 = 3$ where the unique zero z is totally ramified. But in that case τ^2 induces a triple cover of $\mathbb{P}^1$ totally ramified at the Weierstrass point z, which is impossible because the linear system $|3z|$ has a base point at z.

Next consider $b = 3$. By Riemann–Hurwitz $g = 1$ or 2 in this case. For $b = 3$ and $g = 1$, we get $s_1 + s_2 + s_3 = 6$ with $1 \leq s_i \leq 3$ dividing 6. The only possibilities are $(2, 2, 2)$ or $(1, 2, 3)$. The former is impossible for a connected cyclic cover of $\mathbb{P}^1$ as the gcd of s_1, s_2, s_3 is not relatively prime to 6. For the latter, note that the eigenvalues of τ from the absolute homology of X contain ζ and ζ^5. If the concerned stratum for X has any extra dimension (from non-absolute periods or residues), since the determinant of the τ-action is ± 1, the extra eigenvalue(s) together with (ζ, ζ^5) will make $\mathrm{age}(\tau) \geq 1$. If there is no extra dimension, then there is a unique zero and all poles are constrained to have zero residue. Moreover if d polar edges in a special fiber joining d higher connected components get permuted for $d > 2$, then we obtain $\mathrm{age}\,\boldsymbol{\tau} \geq 1$ by (P2), and for $d = 2$ we obtain an extra eigenvalue $-1 = \zeta^3$ which combined with the absolute eigenvalues still makes $\mathrm{age}\,\boldsymbol{\tau} \geq 1$. Therefore, the only remaining case is $\mathbb{P}\Omega\mathcal{M}_1(0)$ where the marked point is totally ramified, which gives **Case (13)**.

For $b = 3$ and $g = 2$, we get $s_1 + s_2 + s_3 = 4$ with $1 \leq s_i \leq 3$ dividing 6. The only possibility is $(1, 1, 2)$. The eigenvalues of τ from the absolute homology of X are (ζ, ζ^5) together with another conjugate pair. Since the target is $\mathbb{P}^1$, there is no invariant absolute period, hence the other pair is either (ζ^2, ζ^4) or (ζ^3, ζ^3). Both cases give $\mathrm{age}(\tau) \geq 1$ after taking projectivization by ζ^a for any a. □

In the sequel we need to bound the number of vertices in a level graph that can be permuted by an automorphism of small age.

Lemma 4.9. *Suppose $\boldsymbol{\tau}$ is an automorphism of a multi-scale differential of type μ (possibly meromorphic) with* $\mathrm{age}(\boldsymbol{\tau}) < 1$. *Then any $\boldsymbol{\tau}$-orbit of cyclicly permuted vertices in the associated level graph has cardinality at most two. Moreover if two vertices are swapped, then $\boldsymbol{\tau}^2$ acts trivially on their underlying stable curves.*

Proof. Suppose there is an orbit of $d > 1$ cyclicly permuted vertices (which have to be on the same level). We need to show that $d = 2$. By Lemma 4.6 $\mathrm{age}(\boldsymbol{\tau})$ is at least one if $d \geq 3, M > 0$ or $d = 2, M > 1$, where recall that M is the dimension of the multi-vertex RC-independent subspace. Therefore, any permuted vertex must have genus zero with at most two zero edges (otherwise $M \geq 2$ from the relative periods of the zeros). Moreover if $d \geq 3$, then each of the permuted vertices can admit only one zero edge (otherwise $M \geq 1$).

Suppose there exists an orbit of $d \geq 3$ permuted vertices. Among all of such orbits we choose the largest d, and further choose the highest level containing the orbit if there are multiple orbits of d permuted vertices. We denote the chosen d permuted vertices by $v_1, \ldots, v_d$. If they admit horizontal edges (in all situations of self-loops, or joining between them, or joining some other vertices on the same level), then there exist at least d independent horizontal plumbing parameters that are cyclicly permuted, making $\operatorname{age}(\boldsymbol{\tau}) \geq 1$ by Lemma 4.6 (using $d \geq 3$ and $M \geq 1$), which contradicts the assumption. We can thus assume that all polar edges of the v_i are vertical only. By the preceding paragraph, each v_i admits only one zero edge and hence at least two vertical polar edges (by stability and since it does not have simple poles). Take a polar edge e_1 of v_1 joining to a higher level and consider part of its $\boldsymbol{\tau}$-orbit $e_1, e_2, \ldots, e_d$ (where the next one e_{d+1} does not have to be e_1 as it might be another polar edge of v_1). Suppose the upper ends of these d edges are adjacent to d' vertices $v'_1, \ldots, v'_{d'}$ that are permuted by $\boldsymbol{\tau}$, where $d = d'k'$. By assumption $d' < d$ (as we chose the d permuted vertices to be both largest and highest) and hence each v'_j admits at least $k' = d/d'$ zero edges joining to some of the v_i. Using $M' \geq k' - 1$ from the relative periods of these zero edges of v'_j, we obtain at least $(d'-1)(k'-1)/2$ for the age by Lemma 4.6. Since $k' = d/d' > 1$, if $d' \geq 3$, then $\operatorname{age}(\boldsymbol{\tau}) \geq 1$ which contradicts the assumption. Hence the only possibilities are $d' = 1$, or $d' = 2$ with $k' = 2$.

Consider first $d' = 1$, i.e. $k' = d$. Then $e_1, \ldots, e_d$ are the zero edges of a single higher-level vertex and they are permuted by $\boldsymbol{\tau}$. Then we can apply (Z) to conclude that $d \leq 4$. The case $d = 4$ is only possible when $e_1, \ldots, e_4$ are the zero edges of a single higher-level vertex in Case (8) of Figure 4, which contributes at least $3/4$ to the age. Since each of the v_i admits at least another polar edge, consider its orbit with $d' = 1$ or $d' = k' = 2$ in the same notation. The former contributes at least another $3/4$ and the latter contributes at least an extra $1/2$, both of which make the total age bigger than one, contradicting the assumption. For the case $d = 3$, the estimate in (Z) gives at least $1/3$. Since each of v_1, v_2, v_3 admits at least another polar edge, consider its orbit with $d' = 1$ in the same notation and we gain another $1/3$ (here $d = 3$ is not divisible by 2, so $d' = 2$ cannot occur). Moreover, v_i cannot admit more than two polar edges (as we have already got $2/3$ for the age). Then the residue cycles r_i of the two polar edges (up to sign) on the v_i can either freely vary or are constrained by $r_1 + r_2 + r_3 = 0$ only, hence the action of $\boldsymbol{\tau}$ on the subspace generated by r_1, r_2, r_3 contains eigenvalues ζ_3, ζ_3^2, contributing an extra $1/3$ and making the total age ≥ 1, leading to a contradiction. Alternatively, the three zero edges of v_1, v_2, v_3 go down to another three permuted vertices (otherwise the adjacent single vertex in lower level contributes at least $1/3$ by checking the cases of $k = 3$ in Figure 4 and Figure 5 which makes the total age ≥ 1). These three new permuted vertices can only admit one zero edge each. Hence we can go down along them again, until we find a single vertex with three permuted edges, which leads to the same contradiction as before.

Consider the remaining possibility that $d' = k' = 2$, and hence $d = 4$. In this case the age contribution from v'_1, v'_2 is already at least $(d'-1)(k'-1)/2 = 1/2$. Since each v_i admits at least another polar edge, consider its orbit and the associated upper adjacent vertices (again with $d' = k' = 2$ as the only possibility). Then we obtain another contribution $1/2$, which altogether makes the total age ≥ 1, leading to a contradiction.

We have thus proved that $d = 2$, i.e., any $\boldsymbol{\tau}$-permuted vertices must appear in pairs. Take two mutually permuted vertices v_1 and v_2. Next we will show that

$\boldsymbol{\tau}^2$ acts trivially on this pair, i.e., $\boldsymbol{\tau}^2$ fixes every edge of v_1 and v_2. Prove by contradiction. First suppose $\boldsymbol{\tau}^2$ does not fix all zero edges in the pair. Then we can find an orbit of zero edges $z_1 \mapsto z_2 \mapsto z_1' \mapsto z_2'$ ($\mapsto z_1$, since we have seen that each permuted vertex can admit at most two zero edges), where z_i and z_i' are the zero edges of v_i for $i = 1, 2$. If these four zero edges belong to a single higher vertex, then it can only be Case (8) in Figure 4 which contributes at least $3/4$ to the age. On the other hand, the permuted pair contributes at least $1/2$ from the relative periods of the zero edges, which altogether makes the total age ≥ 1 and contradicts the assumption. The remaining possibility is that v_i joins a lower vertex v_i' via z_i and z_i', and hence the two banana pairs (v_1, v_1') and (v_2, v_2') are swapped by $\boldsymbol{\tau}$. But the upper relative periods and the lower residues each contribute at least $1/2$, in total making the age ≥ 1 and thus leading to a contradiction. The same argument can be used to show that $\boldsymbol{\tau}^2$ fixes all vertical polar edges of v_1 and v_2. For any horizontal edge of v_i, it cannot be a self-loop at v_i (otherwise its residue cycle and plumbing parameter can make the multi-vertex RC-independent dimension $M \geq 2$). Similarly we can rule out the case that two or more horizontal edges of v_i are permuted by $\boldsymbol{\tau}$. Therefore, every horizontal edge of v_1 joins v_2, which is fixed by $\boldsymbol{\tau}$ (with the two ends swapped). Consequently $\boldsymbol{\tau}^2$ fixes the ends of every horizontal edge at both v_1 and v_2. In summary, we have shown that $\boldsymbol{\tau}^2$ fixes all zero and polar edges on each v_i (whose number is at least three by stability since v_i has genus zero). Hence $\boldsymbol{\tau}^2$ acts trivially on v_1 and v_2. □

Remark 4.10. Suppose v_1 and v_2 are two vertices of genus zero such that $\boldsymbol{\tau}$ swaps them and $\boldsymbol{\tau}^2$ is the identity restricted to each of them. If their relative periods and residues contribute zero to $\mathrm{age}(\boldsymbol{\tau})$, then the above proof implies that each v_i admits a unique zero edge, and moreover, any two permuted (vertical) polar edges must be constrained by the GRC to have the sum of the residues equal to zero. We call such permuted (v_1, v_2) of age zero a *trivial pair*. In particular if we view a trivial pair as a 'single' vertex, then it behaves the same as hyperelliptic vertices of age zero as described in Remark 4.3.

Finally we can show the following result about automorphisms of multi-scale differentials with small age. Recall the coordinates $\mathbb{A}_{\mathrm{rel}}(X)$, $\mathbb{C}_{\mathrm{hor}}$ and $\mathbb{C}_{\mathrm{lev}}$ introduced in Section 2.2, and $\mathbb{A} = \mathbb{A}_{\mathrm{rel}}(X) \times \mathbb{C}_{\mathrm{hor}} \times \mathbb{C}_{\mathrm{lev}}$.

Proposition 4.11. *Let $\boldsymbol{\tau}$ be a lift to $\mathbb{A}$ of an automorphism of a multi-scale differential of type μ (possibly meromorphic). If $\mathrm{age}(\boldsymbol{\tau}) < 1$ and $\boldsymbol{\tau}$ does not induce a trivial action or a quasi-reflection on $\mathbb{A}$, then the action of $\boldsymbol{\tau}$ on the subspace $\mathbb{C}_{\mathrm{lev}}$ of level parameters is non-trivial for all μ in $g \geq 2$ except for Case (3) in $g = 2$ in Figure 4.*

Proof. Suppose that $\boldsymbol{\tau}$ acts trivially on $\mathbb{C}_{\mathrm{lev}}$. We remark that in this case any two $\boldsymbol{\tau}$-fixed vertices connected by vertical edges must have the same $\boldsymbol{\tau}$-restricted order. To see it, let X and Y be two $\boldsymbol{\tau}$-fixed vertices joined by vertical edges $e_1, \ldots, e_d$ that are permuted in one orbit (there could be more edges between them in other orbits). Suppose the $\boldsymbol{\tau}$-orders restricted to X and Y are k_1 and k_2 respectively, and $k_i = d\ell_i$ where ℓ_i is the ramification multiplicity of the quotient map of $\boldsymbol{\tau}$ restricted to each end of the edges. Each e_j gives a relation $x_j y_j$ equal to some product of the level parameters (as in (6)), where x_j and y_j are local standard coordinates at e_j. Note that $\boldsymbol{\tau}^d$ maps x_1 to $\zeta_{\ell_1} x_1$ and maps y_1 to $\zeta_{\ell_2} y_1$ for some primitive ℓ_i-th roots of unity. Since by assumption $\boldsymbol{\tau}$ acts trivially on the level parameters, it implies that $\zeta_{\ell_1}\zeta_{\ell_2} = 1$, and hence $\ell_1 = \ell_2$. Since $k_i = d\ell_i$, we thus conclude that $k_1 = k_2$.

If $\boldsymbol{\tau}$ permutes any vertices, by Lemma 4.9 $\boldsymbol{\tau}^2$ fixes every vertex, acts trivially on all permuted pairs, and acts trivially on the level parameters (since $\boldsymbol{\tau}$ does). Applying the remark in the preceding paragraph to $\boldsymbol{\tau}^2$, it implies that $\boldsymbol{\tau}^2$ is a trivial action. Therefore, $\boldsymbol{\tau}$ has order two and it induces either a trivial action or a quasi-reflection (i.e., with age 0 or 1/2) as the only possibilities of having age smaller than one, which is ruled out by the assumption.

From now on suppose that $\boldsymbol{\tau}$ has order at least three and that $\boldsymbol{\tau}$ fixes every vertex. If there are at least three permuted horizontal edges, then by (SPH1) we get $\operatorname{age}(\boldsymbol{\tau}) \geq 1$. Moreover if a horizontal edge e is fixed at a vertex v, then the local standard form dx/x is invariant at e, hence ω restricted to v is an invariant form, which cannot occur for any cases in Figure 4 and Figure 5, except for possibly hyperelliptic vertices or vertices on which $\boldsymbol{\tau}$ acts trivially. If two horizontal edges (or the two ends of a horizontal self-loop) are permuted, looking at the cases, only hyperelliptic vertices are possible. In summary, horizontal edges are only adjacent to vertices on which the restricted $\boldsymbol{\tau}$-orders are one or two. Combining with the remark that vertical edges join vertices that have the same restricted $\boldsymbol{\tau}$-order, we thus conclude that $\boldsymbol{\tau}$ restricted to every vertex has order exactly k for some $k \geq 3$.

From the Cases of $k \geq 3$ in Figure 4 and Figure 5 we see that the age contribution is at least 1/3 from any $\boldsymbol{\tau}$-fixed vertex of order $k \geq 3$. Since $\operatorname{age}(\boldsymbol{\tau}) < 1$, there can be at most two vertices in the graph. If there is a single vertex, the only case with $g \geq 2$ in Figure 4 is Case (3) in genus two. Suppose there are exactly two vertices v_1 and v_2. Then the only possibilities are Cases (2), (4), (5), (R1), (R2), (R5) for $k = 3$, and Case (13) for $k = 6$. The latter cannot appear twice as its signature does not have a pole. Hence we can assume that $k = 3$. Since all zeros and poles of the ambient stratum are labeled, any permuted zeros and poles in these Cases must be edges. Then Cases (4), (5), (R5) can appear in pairs and Cases (2), (R1), (R2) can appear in pairs (including possibly self-pairing). Due to the congruence conditions and residue conditions in these Cases, the only possibilities are (4) paired with (R5) and (2) paired with (2). In both cases the signatures of the ambient strata belong to Case (3). $\square$

5. Singularities of the coarse moduli space

The purpose of this section is to control the singularities of the coarse moduli space $\mathbb{P}\mathrm{MS}(\mu)$ in order to show that the usual strategy for proving general type—the canonical bundle is ample plus effective—can be used after an appropriate modification due to non-canonical singularities at the boundary as stated in Proposition 1.3, which we will prove at the end of this section.

We start with the interior of the moduli space and the digression on the logarithmic viewpoint:

Theorem 5.1. *For any signature μ, except of type $\mu = (m, 2-m)$ in genus $g = 2$ for $1 \neq m \equiv 1 \bmod 3$ (Case (3) in Figure 4), the interior of the coarse moduli space $\mathbb{P}\mathrm{MS}(\mu)$ has canonical singularities.*

The pair $(\mathbb{P}\mathrm{MS}(\mu), \mathbf{D})$ consisting of the coarse moduli space and the total boundary $\mathbf{D} = \partial\mathbb{P}\mathrm{MS}(\mu)$ is a log canonical pair.

Both statements hold as well for the strata with unlabeled zeros and poles, after further ruling out Cases (6), (7), (8), (11), (12) in Figure 4.

This situation is quite parallel to the moduli space of curves $\overline{M}_g$. The singularities in the interior M_g are also canonical, as shown by [HM82]. The pair $(\overline{M}_g, \partial M_g)$ being log canonical is a general fact for the coarse moduli space of a Deligne–Mumford stack with a normal crossings boundary divisor (see e.g., [HH09, Appendix A]).

The proof of Theorem 5.1 is given in Section 5.2. In contrast, there can be non-canonical singularities at the boundary, see Remark 5.2 for an easy example of such a singularity induced by ghost automorphisms. These are discussed in general in Section 5.3. The effect of curve automorphisms and ghost automorphisms are combined in Section 5.4. There we define the compensation divisor D_{NC} and prove Proposition 1.3.

5.1. **Canonical sheaf and singularities on quotient stacks.** We recall several well-known facts on singularities and the canonical sheaf of an irreducible normal variety W, with a focus on the case of coarse moduli spaces of smooth Deligne–Mumford stacks. In particular, the spaces we consider are $\mathbb{Q}$-factorial, i.e., every Weil divisor is $\mathbb{Q}$-Cartier.

On a singular variety there are three competing definitions of sheaves of differential forms. First, Ω^1_W denotes the sheaf of Kähler differentials and $\Omega^p_W = \wedge^p \Omega^1_W$ its tensor power. It is badly behaved near the singular points and plays hardly any role in the sequel. Second, let $i\colon U := W_{\mathrm{reg}} \to W$ be the inclusion of the regular part and let

$$\Omega^{[p]}_W \;:=\; i_*(\Omega^p_U) = (\Omega^p_W)^{**} \tag{20}$$

be the sheaf of reflexive differential forms. Its top power $\omega_W = \Omega^{[p]}_W$ for $p = \dim(W)$ is a line bundle, called Grothendieck's dualizing sheaf (even though W is not Cohen–Macaulay in general, so Serre duality does not hold with ω_W). Third, let $\pi\colon \widetilde{W} \to W$ be a resolution of singularities and we define

$$\widetilde{\Omega}^p_W \;=\; \pi_*\Omega^p_{\widetilde{W}} \tag{21}$$

which is useful for computing global sections on $\widetilde{W}$. Finally we define the canonical divisor $K_W = \pi_* K_{\widetilde{W}}$ using the pushforward of cycles. Note that $\omega_W = \mathcal{O}(K_W)$.

We now discuss the logarithmic situation, or more generally the case of pairs. Let W still be a normal variety and $\Delta = \sum a_i D_i$ a sum of prime divisors with $a_i \in \mathbb{Q}$. Choose the smooth resolution π so that moreover the preimage $\widetilde{D} = \pi^{-1}D$ is a normal crossings divisor. Now let $i\colon U \to W$ be the inclusion of the open subset where W is smooth and D is normal crossings. As above we define the sheaf of reflexive logarithmic differentials and the pushforward of logarithmic differentials on the resolution to be

$$\Omega^{[p]}_W(\log D) \;:=\; i_*(\Omega^p_U(\log D|_U)) \quad \text{and} \quad \widetilde{\Omega}^p_W(\log D) \;=\; \pi_*\Omega^p_{\widetilde{W}}(\log \widetilde{D})\,. \tag{22}$$

The logarithmic canonical divisor is defined to be $K_W + D$. This is consistent with the above notation since $\mathcal{O}(K_W + D) = \Omega^{[p]}_W(\log D)$ for $p = \dim(W)$.

Next we review some types of singularities of pairs. Recall that the discrepancy $\mathrm{discrep}(W, \Delta)$ is the infimum over all exceptional divisors E in all birational morphisms $\widetilde{W} \to W$ of the coefficient $a(E, \Delta, \widetilde{W})$ of E in the pullback of $K_W + \Delta$. The pair (W, Δ) has *canonical singularities* if $\mathrm{discrep}(W, \Delta) \geq 0$ and *logarithmic canonical singularities* if $\mathrm{discrep}(W, \Delta) \geq -1$. More details can be found in [KM98, Section 2.3].

In particular if $W = (W, \emptyset)$ has canonical singularities, then sections of the canonical bundle restricted to the regular locus of W can be extended across the singularities ([Rei87]).

5.2. **Singularities from curve automorphisms.** We apply this discussion to $W = \mathbb{P}\mathrm{MS}(\mu)$, keep the notation D_Γ for the boundary divisors of the stack and write $\mathbf{D}_\Gamma$ etc for the boundary divisors of the coarse moduli space. Since $\mathbb{P}\mathrm{MS}(\mu)$ has only finite quotient singularities, every subvariety of codimension one is $\mathbb{Q}$-Cartier. Thus the rational Picard groups of $\mathbb{P}\mathrm{MS}(\mu)$ and $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ are identical and computations are performed mostly in terms of the classes of D_Γ in the sequel.

Proof of Theorem 5.1. For the first statement we examine the cases in Figure 4. Except Case (3), all the strata in that table either involve permuted marked points (not allowed for labeled strata), or are one-dimensional with smooth quotients.

To prove the second statement we use that $(\mathbb{P}\mathrm{MS}(\mu), \mathbf{D})$ is the coarse moduli space associated with a smooth Deligne–Mumford stack with normal crossings boundary divisor ([BCGGM2]) and that $\mathbf{D}$ has each boundary term with coefficient one. In fact, [HH09, Proposition A.13] shows that in this situation there is some boundary divisor Δ such that the pullback of $m(K_{\mathbb{P}\mathrm{MS}(\mu)} + \Delta)$ equals $m(K_{\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)} + D)$ and furthermore that $(\mathbb{P}\mathrm{MS}(\mu), \Delta)$ is log canonical. Since all boundary divisors in D appear with coefficient one, the same holds for Δ, i.e., it implies that $\Delta = \mathbf{D}$. This reflects the fact that log canonical divisors are insensitive to branching (see [Fa, Proposition 20.2] for a general comparison formula).

Finally for unlabeled strata we only need to rule out those of (projectivized) dimension at least two in Figure 4. □

Remark 5.2. Consider the left slanted cherry $\Delta = \Delta_\ell$ from Example 3.5, i.e., with the twist $p = 2$ on the short edge. The stack structure is given by the group $K_\Delta = \mathbb{Z}/3\mathbb{Z}$. As we verified there, the generator $(2,1)$ of this group acts by $(\zeta_6^2, \zeta_3^1) = (\zeta_3, \zeta_3)$ on the coordinates corresponding to opening up the levels. (See also Equations (6.7) and (12.6) in [BCGGM2] for the construction. The local parameter called t_i in loc. cit. raised to the lcm of p_e for all edges e crossing the level passage gives the coordinate called s_i that rescales the differential.) This group action does not satisfy the Reid–Tai criterion and [Rei87, Example 1.8 (2)] shows explicitly why the corresponding singularity is not canonical. We will elaborate on this in the next subsection.

Proof of Theorem 1.2. Besides Theorem 5.1 we need to show that non-canonical singularities occur in the boundary of all strata with possible exceptions in low genus only. A more elaborate version of Remark 5.2 can be used to show that as long as $g \geq 5$. Consider a triangle graph with one vertex at each level, the top vertex in the stratum $\Omega\mathcal{M}_{2,2}(0,2)$, the middle vertex in the stratum $\Omega\mathcal{M}_{2,2}(-2,4)$ and the bottom level vertex with the remaining genus and all the marked points. The prongs are 3 on the long edge from the top to bottom level, and 1 and 5 on the short edges. The element $(\zeta_3^1, \zeta_{15}^5)$ fixes all prongs, and it defines a ghost automorphism with age < 1. □

5.3. **Singularities induced by ghost automorphisms.** The singularities induced at the boundary of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$, say at an enhanced level graph $\Gamma \in \mathrm{LG}_L$, stem from the action of the ghost automorphisms $K_\Gamma = \mathrm{Tw}_\Lambda/\mathrm{Tw}_\Lambda^s$. These are toric singularities. We explain here how to fit the data of the graph and the enhancements into the standard framework of toric geometry. The goal is to give a formula for the discrepancies and, more generally, a formula for the pullback of torus-invariant divisors in terms of these graph data.

We start by recalling some well-known toric terminologies. An affine toric variety is given by a $\mathbb{Z}$-module N that we view as a lattice inside $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and a convex rational polyhedral cone $\sigma \subset N_\mathbb{R}$. We let $M = N^\vee = \mathrm{Hom}(N, \mathbb{Z})$ and view the dual cone $\sigma^\vee$ as a subset of $M_\mathbb{R}$. Then the group algebra $\mathbb{C}[M \cap \sigma^\vee]$ is a finitely generated algebra and the associated (affine) toric variety is defined as

$$X_{N,\sigma} \;=\; \mathrm{Spec}(\mathbb{C}[M \cap \sigma^\vee])\,.$$

The spanning rays of σ generated by the primitive elements $r_1, \ldots, r_L$ are in bijection to the torus-invariant divisors $D_1, \ldots, D_L$ of $X_{N,\sigma}$. We omit σ from the notation, if it is the positive cone for some implicitly chosen basis of $N_\mathbb{R}$.

The affine toric variety X_N is non-singular if $\sigma \cap N$ is generated by a subset of a basis of N. If this is not the case, we can resolve the singularities by subdividing σ as a union of subcones such that each of the subcones satisfies the above condition. Let F be the fan obtained from the cone subdivision. The additional rays of the subcones are given by the primitive interior points in $\sigma^\circ \cap N$, which we list as $v_1, \ldots, v_s$. Each of the rays v_i corresponds to a torus-invariant exceptional divisor E_i in the resolution $\pi\colon \widetilde{X}_F \to X_N$.

We state the next proposition in the case of interest to us, namely that X_N has only abelian finite quotient singularities, which is equivalent to σ being a simplicial cone by [CLS11, Theorem 11.4.8], i.e., $L = \dim N_\mathbb{R}$. Consequently there are elements $m_{\sigma,i} \in M_\mathbb{R}$ such that $\langle m_{\sigma,i}, r_j \rangle = \delta_{ij}$. Let $m_\sigma = \sum_{i=1}^L m_{\sigma,i}$. We denote the non-exceptional torus-invariant divisors, the strict transforms of the D_i by $\widetilde{D}_i$.

Proposition 5.3. *The canonical divisor $K_{X_N} = -\sum_{i=1}^L D_i$ is the negative sum of the torus-invariant divisors. Moreover if X_N has only abelian quotient singularities, then the discrepancy of E_i is given by*

$$K_{\widetilde{X}_F} - \pi^* K_{X_N} \;=\; \sum_{j=1}^{s} (\langle m_\sigma, v_j \rangle - 1) E_j$$

and more generally

$$\pi^* D_i - \widetilde{D}_i \;=\; \sum_{j=1}^{s} \langle m_{\sigma,i}, v_j \rangle E_j \quad \textit{for all} \quad i = 1, \ldots, L\,.$$

Proof. The claims follow from combining [CLS11] Theorem 8.2.3 on the description of the canonical bundle, Lemma 11.4.10 for its pullback to resolutions, Theorem 4.2.8 for the conversion of divisors into support functions and Proposition 6.2.7 for the pullback of divisors written in these terms. □

In general, not all pluricanonical forms extend from X_N to its resolution $\widetilde{X}_F$, since they can acquire poles along the exceptional divisors E_i. However, we can consider only a subset of pluricanonical forms having high enough order of vanishing along the divisors D_i. We now show a criterion about how high the order of vanishing of pluricanonical forms along D_i has to be in order to ensure that they extend to the resolution.

Proposition 5.4. *Let $(b_1, \dots, b_L) \in \mathbb{N}^L$ be a tuple such that*

$$\sum_{i=1}^{L} (b_i + 1)\langle m_{\sigma,i}, v_j \rangle \geq 1, \quad \textit{for } j = 1, \dots, s. \tag{23}$$

Then for all $a \in \mathbb{N}$ we have the inequality

$$h^0(\widetilde{X}_F, aK_{\widetilde{X}_F}) \ \geq \ h^0\Big(X_N, a\Big(K_{X_N} - \sum_{i=1}^{L} b_i D_i\Big)\Big)\,.$$

Proof. It suffices to show that the exceptional divisors E_j occur with non-negative coefficients in the difference

$$\begin{aligned} &K_{\widetilde{X}_F} \ - \ \pi^*\Big(K_{X_N} - \sum_{i=1}^{L} b_i Di\Big) \\ &= \ \sum_{i=1}^{L} b_i \widetilde{D}_i \ + \ \sum_{j=1}^{s} (\langle m_\sigma, v_j\rangle - 1)E_j \ + \ \sum_{j=1}^{s} E_j \cdot \sum_{i=1}^{L} b_i \langle m_{\sigma,i}, v_j\rangle \\ &= \ \sum_{i=1}^{L} b_i \widetilde{D}_i \ + \ \sum_{j=1}^{s} E_j \cdot \Big(-1 + \sum_{i=1}^{L} (b_i+1)\langle m_{\sigma,i}, v_j\rangle\Big)\,, \end{aligned}$$

which is ensured by the standing assumption. ☐

Recall that an inclusion of lattices $N' \hookrightarrow N$ with quotient group G, and the same cone σ in both lattices, gives rise to a quotient map $X_{N'} \mapsto X_{N'}/G = X_N$, see e.g. [CLS11, Sections 1.5 and 3.3].

We focus now on toric varieties obtained by the the quotient of affine space via a cyclic group of order n. We say that a singularity is of type $\frac{1}{n}(a_1, \dots, a_L)$ if it is the quotient of $\mathbb{C}^L$ by a cyclic group $G = \langle \tau \rangle$ of order n acting by $\zeta_n^{a_i}$ on the L coordinates. Consider then the case $X_{N'} = \mathbb{C}^L$, so N' is a lattice in $\mathbb{R}^L$ generated by vectors e_i and σ is the standard cone generated by the basis (e_i) of N'. If we define N to be the lattice generated by the basis of N' and by $v_\tau = \sum_{i=1}^{L} a_i/n \cdot e_i$, then $N/N' = G$ and $X_N = \mathbb{C}^L/G$.

We specialize further to the toric varieties X_N. Since the cone σ is generated by the coordinate vectors $e_i = v_i$, using the notation previously introduced, the $m_{\sigma,i}$ are simply the dual vector $e_i^\vee$. The only primitive interior points in $\sigma^\circ \cap N$ are the vectors v_{τ^j}, for $j = 1, \dots, n$. Hence, in this setting, Proposition 5.4 specializes to the following statement.

Corollary 5.5. *Let $X = \mathbb{C}^d/\langle \tau \rangle$, where τ acts by multiplication of $\zeta_n^{a_i}$ on the i-th coordinate. If*

$$\sum_{i=1}^{L} \frac{a_i}{n}(b_i + 1) \geq 1 \tag{24}$$

then the inequality

$$h^0(\widetilde{X}, aK_{\widetilde{X}}) \ \geq \ h^0\Big(X, a\Big(K_X - \sum_{i=1}^{d} b_i D_i\Big)\Big)$$

holds for all $a \in \mathbb{N}$, where D_i are the image in X of the divisors $\{x_i = 0\} \subset \mathbb{C}^d$ and where $\widetilde{X}$ is a smooth resolution of X.

Note that if the age of τ is indeed greater or equal to one, then we take $b_i = 0$, and so the singularities of X_N are canonical.

Example 5.6. The resolution $\pi\colon \widetilde{X} \to X$ of a singularity of type $\frac{1}{3}(1,1)$ has a single exceptional divisor E. In this case we have

$$K_{\widetilde{X}} - \pi^* K_X \;=\; -\frac{1}{3}E \quad \text{and} \quad \pi^* D_i - \widetilde{D}_i \;=\; \frac{1}{3}E \tag{25}$$

where D_i for $i = 1, 2$ are the two coordinate axes. If we consider for example $b_1 = 1$ and $b_2 = 0$, then by (24) we have that all sections of $K_X - D_1$ extend to the resolution $\widetilde{X}$.

The resolution $\pi\colon \widetilde{X} \to X$ of a singularity of type $\frac{1}{4}(1,2)$ also has a single exceptional divisor E. In this case we have

$$K_{\widetilde{X}} - \pi^* K_X \;=\; -\frac{1}{4}E \quad \text{and} \quad \pi^* D_1 - \widetilde{D}_1 \;=\; \frac{1}{4}E, \quad \pi^* D_2 - \widetilde{D}_2 \;=\; \frac{1}{2}E\,. \tag{26}$$

In this case, if we set for example $b_1 = 1$ and $b_2 = 0$, by (24) we have that all sections of $K_X - D_1$ extend to the resolution $\widetilde{X}$. This is a special case of the resolutions in Example 5.7.

Even though not needed in the sequel, it is instructive to describe in standard toric geometry language the structure near a boundary component of the orderly blow-up $\mathbb{P}\mathcal{MS}(\mu)$, where only ghost automorphisms are quotiented out. Recall indeed that the map $\varphi_1\colon \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) \to \mathbb{P}\mathcal{MS}(\mu)$ to the orderly blowup is locally given by the map $[U/K_\Gamma] \to U/K_\Gamma$, where U is a neighborhood of a generic point of a boundary component D_Γ and $K_\Gamma = \mathrm{Tw}_\Gamma/\mathrm{Tw}_\Gamma^s$ is the group of ghost automorphisms.

Recall from Section 2.2 the coordinate system near the boundary. We analyze the toric geometry of the part $\mathbb{C}^{\mathrm{lev}}$ of this coordinate system. This is the affine toric variety with the (dual) lattice

$$\begin{aligned} M' \;&=\; \Big\langle \frac{1}{p_e}\cdot w_i, \quad e \in E(\Gamma),\, i\text{-th level passage crossed by } e \Big\rangle_{\mathbb{Z}} \\ &=\; \Big\langle \frac{1}{\ell_i}\cdot w_i, \quad i = 1,\dots,L \Big\rangle_{\mathbb{Z}} \end{aligned}$$

and $\sigma^\vee$ the positive (dual) cone generated by the w_i in $M'_{\mathbb{R}} \cong \mathbb{R}^L$, where w_i is the i-th unit vector and ℓ_i is the lcm of all enhancements p_e crossing the i-th level passage. Moreover,

$$N' \;=\; (M')^\vee \;=\; \mathrm{Tw}_\Gamma^s \;=\; \langle \ell_i \cdot e_i,\, i = 1,\dots,L\rangle_{\mathbb{Z}} \tag{27}$$

is the simple twist group by definition, where e_i is the dual vector of w_i.

In order to define the twist group similarly, recall from [BCGGM2, Section 5] that it depends only on the level passages crossed by the edges and the enhancements, not on the vertices the edges are attached to. For $0 \le i < j \le L$ we let

$$w_{-j}^{-i} \;=\; (0,\dots,0,1,\dots,1,0,\dots,0) \in \mathbb{R}^L$$

be the vector where the string of ones goes from $i+1$ to j. Then

$$N := \mathrm{Tw}_\Gamma \;=\; (M')^\vee \quad \text{and} \quad M \;=\; \Big\langle \frac{1}{p_e}\cdot w_{e^-}^{e^+}, \quad e \in E(\Gamma) \Big\rangle_{\mathbb{Z}}$$

where $e^{\pm}$ are the upper and lower ends of e. Note that the explicit computation of a basis of M is in general not possible in closed form, which requires working with gcds, i.e., computing a Smith normal form.

Example 5.7. We continue with the running example of the cherry graph, now generalizing to enhancements a on the short edge and b on the long edge. We let $\ell_1 = \mathrm{lcm}(a,b)$ and $\ell_2 = b$. Then

$$\begin{aligned} N' &= \mathrm{Tw}^s_\Gamma = \Big\langle (\frac{1}{a},0),(\frac{1}{b},0),(0,\frac{1}{b})\Big\rangle^\vee = \Big\langle (\ell_1,0),(0,\ell_2)\Big\rangle \\ N &= \mathrm{Tw}_\Gamma = \Big\langle (\frac{1}{a},0),(\frac{1}{b},\frac{1}{b})\Big\rangle^\vee = \Big\langle (a,-a),(0,\ell_2)\Big\rangle \end{aligned}$$

so that $n := [N : N'] = \ell_1/a = b/\gcd(a,b)$.

We restrict moreover to $b \geq a$. We see that this is a cyclic quotient singularity of order $n = \ell_1/a$ and of type $\frac{1}{n}(1,q)$ where $q = \frac{b-a}{\gcd(a,b)}$. This generalizes Remark 5.2. To resolve this singularity minimally we have to insert the rays generated by boundary points of the lower convex hull of N in the positive quadrant $\sigma = \sigma'$. These are the rays generated by

$$v_j = j \cdot (\tfrac{1}{n}, \tfrac{q}{n}) \in \sigma^\circ \cap N$$

for $j = 1, \ldots, q'$ with $0 < q' < n$ and $qq' \equiv 1 \bmod n$, see [Gee88, Section II.6] or [CLS11, Section 10] for another version of this resolution ('Hirzebruch-Jung continued fraction').

5.4. **Singularities at the boundary.** Here we combine the previous two subsections to analyze the singularities at the boundary with the goal of proving Proposition 1.3. We start with the definition of the non-canonical compensation divisor D_{NC}.

First we distinguish several special edge types in a two-level graph (for vertical edges only). If an edge corresponds to a separating node, we say that it is of *compact type (CPT)*. Otherwise we say that it is of *non-compact type (NCT)*. If the lower part of the graph separated by a CPT edge consists of a single rational vertex, the edge type is called a *rational bottom tail (RBT)*. Recall that a *(vertical) dumbbell (VDB)* graph is defined to be a graph of compact type with a unique (separating) edge which is vertical. If the graph contains a unique (vertical) edge (i.e., a VDB graph) and if one end of the edge is of genus one, we say that it is an *elliptic dumbbell (EDB)*. An edge of compact type which is neither RBT nor EDB is called *other compact type (OCT)*.

Let E_Γ be the number of (vertical) edges of a two-level graph Γ. We then define (for $g \geq 2$) that

$$\begin{aligned} D_{\mathrm{NC}} &:= \sum_{\Gamma \in \mathrm{LG}_1} b^\Gamma_{\mathrm{NC}}[D_\Gamma] := \sum_{\Gamma \in \mathrm{LG}_1} (\ell_\Gamma R^\Gamma_{\mathrm{NC}} - 1)[D_\Gamma] \quad \text{where} \\ R^\Gamma_{\mathrm{NC}} &= \sum_{\mathrm{NCT}} \frac{1}{2}\frac{1}{p_e} + \sum_{\mathrm{RBT}} \frac{1}{p_e} + \sum_{\mathrm{OCT}} \frac{2}{p_e} + \sum_{\mathrm{EDB}} \frac{4}{p_e}\,. \end{aligned} \tag{28}$$

In the above each sum runs over the edges of the corresponding type. Note that the edge type EDB is exclusive, i.e., if it appears in a two-level graph, then the graph has a unique edge and all other edge types do not appear. In particular, an edge cannot be both RBT and EDB due to the assumption that $g \geq 2$.

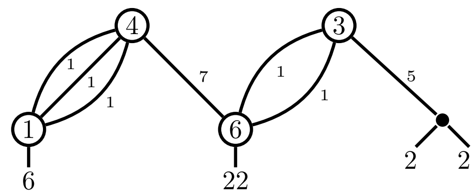


FIGURE 6. In this graph there are five NCT edges with prong 1, one OCT edge with prong 7, and one RBT edge with prong 5.

Remark 5.8. For certain range of genera and signatures, one can alter the definition of R^Γ_{NC} in (28) slightly. Indeed for certifying general type for the minimal strata in low genera, we need another version of R^Γ_{NC} (see Proposition 5.12 and Proposition 5.13).

We are now ready to present the main proof of this section.

Proof of Proposition 1.3. The content of the proposition is that global sections of $a(K_{\mathbb{P}\mathrm{MS}(\mu)} - D_{\mathrm{NC}})$ extend to a-canonical sections on a smooth resolution $\widetilde{\mathbb{P}\mathrm{MS}}(\mu)$ of $\mathbb{P}\mathrm{MS}(\mu)$, for any $a \in \mathbb{N}$. We revisit the argument [Tai82, Proposition 3.1] for this purpose.

Suppose such a section η does not extend to $\widetilde{\mathbb{P}\mathrm{MS}}(\mu)$ and suppose this happens near some multi-scale differential $(X, \boldsymbol{\omega}, \mathbf{z}, \boldsymbol{\sigma})$ compatible with some level graph Π, in fact necessarily in the boundary by Theorem 5.1, i.e. Π is non-trivial The section η thus acquires poles near a divisor E of $\widetilde{\mathbb{P}\mathrm{MS}}(\mu)$. Using the same notation as in Section 2.2, we consider the local covering $\mathbb{A}$ of $\mathbb{P}\mathrm{MS}(\mu)$, where $\mathbb{A} = \mathbb{C}_{\mathrm{lev}} \times \mathbb{A}_{\mathrm{rel}}(X) \times \mathbb{C}_{\mathrm{hor}}$, the tangent space to the orbifold chart near $(X, \boldsymbol{\omega}, \mathbf{z}, \boldsymbol{\sigma})$. Consider finally the normalization of $\widetilde{\mathbb{P}\mathrm{MS}}(\mu)$ in the function field of $\mathbb{A}$, i.e., the normalization of the corresponding fiber product. For each component E' of the preimage of E in this normalization, the stabilizer (in the full Deck group of the cover, the extension $\mathrm{Iso}(X, \boldsymbol{\omega})$ of $\mathrm{Aut}(X, \boldsymbol{\omega})$ by K_Π as in Remark 2.1) is cyclic, say generated by an element $\boldsymbol{\tau}$, a lift of an automorphism in $\mathrm{Aut}(X, \boldsymbol{\omega})$ to an automorphism acting on $\mathbb{A}$ as considered in Section 4. Consequently the pullback of η to $\mathbb{A}/\langle\boldsymbol{\tau}\rangle$ does not extend to its smooth resolution. We will show that this does not happen for sections under consideration, i.e., with enough vanishing along some boundary divisors, using Corollary 5.5.

If $\boldsymbol{\tau}$ is a quasi-reflection on $\mathbb{A}$, the quotient is smooth and the extension of a-canonical sections is automatic. In the case of $\mathrm{age}(\boldsymbol{\tau}) \geq 1$, the singularities of $\mathbb{A}/\langle\boldsymbol{\tau}\rangle$ are canonical and all a-canonical sections of $K_{\mathbb{P}\mathrm{MS}(\mu)}$ extend to $\mathbb{A}/\langle\boldsymbol{\tau}\rangle$ by the original argument of Tai's criterion.

In the remaining cases, by Proposition 4.11 and as our genus and signature hypothesis, $\boldsymbol{\tau}$ acts non-trivially on at least one of the level coordinates t_i. We may thus assume that $\boldsymbol{\tau}$ acts by $\exp(2\pi i a_j/k)$ in an appropriate basis of $\mathbb{A}_{\mathrm{rel}}(X) \times \mathbb{C}_{\mathrm{hor}}$ and by $\exp(2\pi i c_j/k_j)$ on the coordinates t_j, where at least one of the entries c_i of $n_{\boldsymbol{\tau}}$ is non-zero by our assumption. Using Corollary 5.5 with $b_i = b_{\mathrm{NC}}^{\delta_i(\Gamma)}$ for $i = 1, \ldots, L$ (where $\delta_j(\Gamma) \in \mathrm{LG}_1$ is the undegeneration of a level graph Γ obtained by compressing all level passages but the j-th one) and with $b_i = 0$ for $i = L+1, \ldots, d$, we need to show that

$$c(\boldsymbol{\tau}) := \sum_{i=1}^{L} \frac{c_j}{k_j}\left(b_{\rm NC}^{\delta_i(\Gamma)}+1\right)+\sum_{i=L+1}^{d}\frac{a_j}{k}\geq 1\,. \tag{29}$$

This is the statement of Proposition 5.11 below, which is long and technical so we separate it. □

It remains to introduce and justify Proposition 5.11 used in the above proof. This requires some additional preparation. Denote by

$$0\leq s_j \;=\; c_j/k_j<1$$

the (rational) argument of the action of $\boldsymbol{\tau}$ on t_j (mod $2\pi i$) as used in (29). For a (vertical) edge e, denote by $[e]$ the interval of level passages crossed by e. We say that a level passage is *non-trivial* if the corresponding $s_j>0$ and that an edge is *non-trivial* if it crosses a non-trivial level passage. We also say that a vertex has *order* k if the order of $\boldsymbol{\tau}$ restricted to that vertex has order k. Finally for an edge e we define its *contribution*

$$c_e \;=\; \sum_{j\in[e]}(\ell_j/p_e)s_j$$

which depends on $\boldsymbol{\tau}$ but we skip it in the notation when the context is clear.

Lemma 5.9. *Let e be an edge fixed by $\boldsymbol{\tau}$ and joining two vertices v_1 and v_2 where each v_i has order k_i. Suppose either $k_1\neq k_2$, or $k_1=k_2$ and e is non-trivial. Then* $\operatorname{lcm}(k_1,k_2)c_e$ *is a positive integer. In particular,*

$$c_e \;\geq\; \frac{1}{\operatorname{lcm}(k_1,k_2)}\,. \tag{30}$$

Proof. Recall the local equation $x_1x_2=\prod_{j\in[e]}t_j^{\ell_j/p_e}$ at the node represented by e in the universal family over the moduli space of multi-scale differentials, where x_i is a standard coordinate at e in v_i. Consider first the case $k_1\neq k_2$. Since $\boldsymbol{\tau}(x_i)=\zeta_{k_i}x_i$ where ζ_{k_i} is a primitive k_i-th root of unity, $\boldsymbol{\tau}(x_1x_2)$ differs from x_1x_2 by a non-trivial root of unity of order at most $\operatorname{lcm}(k_1,k_2)$. Hence in this case $c_e\geq 1/\operatorname{lcm}(k_1,k_2)$. If $k_1=k_2=k$, then $\boldsymbol{\tau}(x_1x_2)$ differs from x_1x_2 by a root of unity of order at most k, which implies that c_e is either zero or at least $1/k$. However the former is impossible since by assumption e crosses some non-trivial level passage, i.e., at least some $s_j>0$ in the sum. This thus verifies the inequality (30). □

Lemma 5.10. *Let $e_1,\dots,e_h$ be edges which are cyclically permuted by $\boldsymbol{\tau}$ and which join two vertices v_1 and v_2 fixed by $\boldsymbol{\tau}^h$, where each v_i has order k_i. Suppose either $k_1\neq k_2$, or $k_1=k_2$ and the e_i are non-trivial. Then* $\operatorname{lcm}(k_1,k_2)c_{e_i}$ *is a positive integer. In particular*

$$c_{e_i}\geq\frac{1}{\operatorname{lcm}(k_1,k_2)} \tag{31}$$

for all i.

Proof. Note that $\boldsymbol{\tau}^h$ fixes each e_i and the two vertices, and that it has order k_i/h on each v_i. Then the proof of the previous lemma implies that hc_e is a multiple of $1/\operatorname{lcm}(k_1/h,k_2/h)$. It is in fact a non-zero multiple by the previous proof in the case of $k_1\neq k_2$, and directly by the existence of a non-trivial level passage in the case of $k_1=k_2$. □

Finally we let

$$r_j = b_{\mathrm{NC}}^{\delta_j(\Gamma)} + 1$$

and rewrite the contribution $c(\boldsymbol{\tau})$ in (29) in terms of the above notations as

$$c(\boldsymbol{\tau}) \;=\; \sum_{j=1}^{L} r_j s_j + \operatorname{age}(\boldsymbol{\tau})|_{\mathbb{A}_{\mathrm{rel}}\times\mathbb{C}_{\mathrm{hor}}}$$

where by definition $\operatorname{age}(\boldsymbol{\tau})|_{\mathbb{A}_{\mathrm{rel}}\times\mathbb{C}_{\mathrm{hor}}} = \sum_{i=L+1}^{d} \frac{a_j}{k}$.

Proposition 5.11. *For $g \geq 2$, suppose that $\boldsymbol{\tau}$ does not induce a quasi-reflection on $\mathbb{A}$ and that not all level passages are trivial under $\boldsymbol{\tau}$. Then $c(\boldsymbol{\tau}) \geq 1$.*

Before showing the proof of the above proposition, we present an alternative version of R_{NC}^{Γ} which we will need to use to prove Theorem 1.4 in low genera. (More precisely, we will need the following version of R_{NC}^{Γ}, together with the improvement in Proposition 5.13, to show that the minimal strata with odd spin parity are of general type for $13 \leq g \leq 43$.)

Proposition 5.12. *For $g \geq 2$, suppose that $\boldsymbol{\tau}$ does not induce a quasi-reflection on $\mathbb{A}$ and that not all level passages are trivial under $\boldsymbol{\tau}$. Let $v^{\top}$ be the number of top level vertices in Γ. Then substituting R_{NC}^{Γ} in the definition (28) with*

$$R_{\mathrm{NC}}^{\Gamma} = \sum_{\mathrm{NCT}} \frac{1}{E_\Gamma}\frac{1}{p_e} + \sum_{\mathrm{RBT}} \frac{1}{p_e} + \sum_{\mathrm{OCT}} \frac{2}{p_e} + \sum_{\mathrm{EDB}} \frac{4}{p_e} + (v^{\top} - 1)$$

still satisfies that $c(\boldsymbol{\tau}) \geq 1$.

Our strategy is to prove first Proposition 5.12. Since $1/E_\Gamma \leq 1/2$ in the presence of NCT edges, the same proof will work for Proposition 5.11 if we can show that the additional $v^{\top}$-term, which is not present in (28), is not needed if the coefficient for NCT edges is $1/2$ instead of $1/E_\Gamma$.

Proof of Proposition 5.12. We denote by Π the level graph on which we perform the analysis. If $\boldsymbol{\tau}$ permutes some vertices of Π, then $\operatorname{age}(\boldsymbol{\tau})|_{\mathbb{A}_{\mathrm{rel}}\times\mathbb{C}_{\mathrm{hor}}} \geq 1$ unless the permuted vertices consist of two vertices of genus zero swapped by $\boldsymbol{\tau}$ as described in Lemma 4.9. For such a permuted pair we can combine the two vertices as one 'hyperelliptic' vertex, which does not affect the analysis of edge contributions when we apply Lemma 5.10 (for $h = 2$). Moreover, a trivial pair of age zero described in Remark 4.10 behaves the same way as a single hyperelliptic vertex of age zero. In this sense from now on we can assume that $\boldsymbol{\tau}$ *fixes every vertex of* Π. Moreover, we can also assume that Π *has no horizontal edges* (since higher order vertices with age < 1 do not admit such edges according to the tables and as a consequence of the trivial pair discussion in the proof of Proposition 5.11).

Let H be a non-trivial level passage of Π such that it is crossed by the maximum number of edges among all non-trivial level passages, where we denote by E_H the number of edges crossing H. By (30) or (31), any non-trivial edge joining two vertices of order k_1 and k_2 contributes at least $c_e \geq 1/\operatorname{lcm}(k_1,k_2)$ times the corresponding (ℓ/p_e)-coefficient in (28), which is then at least $1/(E_{\mathrm{H}} \operatorname{lcm}(k_1,k_2))$ for all edge types. Hence we can sum up the contributions of the E_{H} edges and obtain that

$$c(\boldsymbol{\tau}) \geq \frac{E_{\mathrm{H}}}{E_{\mathrm{H}} \operatorname{lcm}(k_1,k_2)} = \frac{1}{\operatorname{lcm}(k_1,k_2)}\,. \tag{32}$$

If $k_1 = k_2 = 1$ for these edges, then we obtain enough contribution to $\operatorname{age}(\boldsymbol{\tau})$.

Another preliminary remark is that, if there is a level passage j_0 whose corresponding two-level graph has $v^\top = v^\top_{j_0} > 1$ vertices on top level, then there is a *special edge* e crossing this level passage such that all the level passages $j \in [e]$ satisfy $v^\top_j > 1$, where $v^\top_j$ denotes the number of top level vertices of $\delta_j(\Pi)$. To see the existence of such a special edge e, consider all non-backtracking paths that start and end with an edge crossing j_0, with exactly these two edges crossing j_0, and that connect two disjoint connected components above j_0. By hypothesis this set is nonempty, and we can orient the paths such that the starting level of the paths is not above the ending level. This means that the first edge e_1 of the path has the property that all level passages in $[e_1]$ above j_0 satisfy $v^\top_j > 1$. Consider now a path where the lowest level touched by the path is as high as possible among all paths. Then the first edge e_1 of this path also has the property that all level passages in $j \in [e_1]$ satisfy $v^\top_j > 1$. Indeed if there is a level passage $j' \in [e_1]$ below j_0 with $v_{j'} = 1$, then we would find another path with j' as the lowest touched level. In summary, one can use $e = e_1$ as a special edge. Note that if the starting level passage j_0 is non-trivial, e.g., $j_0 = H$, then the special edge e is non-trivial, since it crosses j_0. Hence by Lemma 5.9 or Lemma 5.10, we obtain a contribution of at least

$$(c_e/E_{\mathrm{H}} + 1)/\operatorname{lcm}(k_1, k_2) \tag{33}$$

where k_i are the order of the vertices joined by e. We call this the $v^\top$*-contribution* of the special edge.

From now on we can assume that there is at least one vertex of order $k_i > 1$. In this case the edge contribution can become smaller, but the vertex age can make an extra contribution to $c(\boldsymbol{\tau})$ by using Figure 4 and Figure 5. We thus need to analyze a number of cases depending on the orders of vertices.

Case that all vertices are of order one or two. Assume that all vertices are of order one or two, and there is at least one vertex of order two. For any such vertex, we can assume that it is hyperelliptic under $\boldsymbol{\tau}$ (otherwise the vertex age is at least one). In this case an edge is either fixed or permuted with another edge under the hyperelliptic involution. Since the total edge contribution is at least $1/2$ from the contribution (32) (for $k_1, k_2 \in \{1, 2\}$), we can assume that the age of every hyperelliptic vertex is zero, since $\operatorname{age}(\boldsymbol{\tau}) \geq 1$ and we are done otherwise. By Remark 4.3 and Lemma 4.7, a hyperelliptic vertex of age zero (including the age from $\mathbb{C}_{\mathrm{hor}}$) does not admit horizontal edges, has at most one fixed zero or one pair of permuted zeros, has every fixed pole with residue zero constrained by GRC, and has every permuted pair of poles with the zero sum of the residues constrained by GRC. Therefore, we can assume that every hyperelliptic vertex belongs to a hyperelliptic (banana) tree of age zero, which is a connected subgraph consisting of hyperelliptic vertices of age zero whose edge configurations are described above. If a non-hyperelliptic vertex is adjacent to one end of a hyperelliptic tree, we say that the connecting edge is a *handle*.

Among the E_{H} edges crossing H, suppose E_1 edges are fixed and have both endpoints trivial (*type one*), E_2 edges are fixed and have at least one endpoint hyperelliptic (*type two*), and $2E_3$ edges are permuted in pairs with both endpoints hyperelliptic (*type three*), where $E_1 + E_2 + 2E_3 = E_{\mathrm{H}}$. By (30) and (31), edges e_1, e_2, (e_3, e_3') of the three types satisfy that $c_{e_1} \geq 1$, $c_{e_2} \geq 1/2$, and $c_{e_3} + c_{e_3'} \geq 1/2 + 1/2 = 1$, respectively. If there are edges of type two or three constrained by

GRC, then the undegeneration of the level passage H gives rise to a two-level graph with $v_{\mathrm{H}}^{\top} > 1$, and the $v^{\top}$-contribution (33) is at least $1/2$ which is enough. Hence we can assume that for every edge of type two which is not a handle, or for every pair of edges of type three, there is at least one associated CPT handle or at least two NCT handles. Note that a handle e of a hyperelliptic tree has endpoints of order one and two respectively, hence $c_e \geq 1/2$. In addition, any permuted edge of type three is not a handle.

Consider first the case that every handle of a hyperelliptic tree is not CPT. Then every hyperelliptic tree has at least two handles, one at each end. It follows that $c(\boldsymbol{\tau}) \geq E_1/E_{\mathrm{H}} + (1/2+1/2)E_2/E_{\mathrm{H}} + (1+1/2+1/2)E_3/E_{\mathrm{H}} = 1$. Next consider the case that there are at least two CPT handles. Then $c(\boldsymbol{\tau}) \geq 1/2 + 1/2 = 1$. Finally suppose there is exactly one CPT handle e. Then $c(\boldsymbol{\tau}) \geq 1/2 + E_1/E_{\mathrm{H}} + E_2/E_{\mathrm{H}} + 2E_3/E_{\mathrm{H}} - 1/E_{\mathrm{H}} \geq 1$ for $E_{\mathrm{H}} \geq 2$, where we subtract $1/E_{\mathrm{H}}$ for not counting redundantly the contribution from the hyperelliptic tree with the handle e. If $E_{\mathrm{H}} = 1$, then the graph consists of a single hyperelliptic tree (modulo trivial level passages), hence either $c(\boldsymbol{\tau}) \geq 1$ or it induces a quasi-reflection.

From now on we assume that besides vertices of order one and two, all other vertices are of order three, four, or six from Figure 4 and Figure 5.

Case that a vertex is of order four. We start by assuming that there is at least one vertex of order four. Since all vertices of order four have age at least $1/2$, we can assume that there is a unique vertex v of order four, which rules out Case (8) as it has four cyclicly permuted edges and thus requires another vertex of the same type to pair with it, giving enough vertex age. If v has a pair of permuted edges, then they can only join a hyperelliptic vertex of age zero. Since $\mathrm{age}(v) = 3/4$ in the relevant Cases (11), (12), (R6) and (R7), we can assume that all other vertices are of order one or two. As before, since the E_{H} edges contribute at least $1/4$, we conclude that $c(\boldsymbol{\tau}) \geq \mathrm{age}(v) + 1/4 = 1$.

Now suppose all edges of v are fixed. The relevant cases are (9) and (10), for which $\mathrm{age}(v) = 1/2$. Case (9) has genus one with a unique edge e going down. Suppose the lower end v' of e has order k' which can be one, two, or three (as in Case (13) of order 6 cannot be a lower vertex). Since e is CPT (and not RBT as $g \geq 2$), we obtain that $c(\boldsymbol{\tau}) \geq \mathrm{age}(v) + \mathrm{age}(v') + 2/\operatorname{lcm}(4, k') \geq 1$ for any $k' = 1, 2, 3$.

For Case (10), if it admits one edge and one leg, then the same argument works as in Case (9). Suppose it admits two edges e_1 and e_2, one going up to v_1 and the other going down to v_2 where v_i is of order $k_i \neq 4$. As before we can reduce to the case that all non-trivial edges are not CPT, any hyperelliptic vertex has age zero, and there is at most one other vertex with non-zero age. Note that e_1 and e_2 cross different level passages. If there is no vertex of order three or six, then $c(\boldsymbol{\tau}) \geq \mathrm{age}(v) + (1/4 + 1/4 + (E_{\mathrm{H}} - 1)/2)/E_{\mathrm{H}} = 1$. If there is exactly one vertex of order three or six, then $c(\boldsymbol{\tau}) \geq \mathrm{age}(v) + 1/3 + (1/12 + 1/4 + (E_{\mathrm{H}} - 1)/6)/E_{\mathrm{H}} \geq 1$.

Case that a vertex is of order six. Now we assume that all vertices are of order one, two, three, or six, and there is at least one of order six. Consider Case (13) for a vertex v of order six where $\mathrm{age}(v) = 1/3$ and v admits a unique edge e going down. Let v' be the lower end of e with order k'. Since e is CPT (and not RBT as $g \geq 2$), its contribution to $c(\boldsymbol{\tau})$ is $2c_e \geq 2/\operatorname{lcm}(6, k') \geq 1/3$, hence combining with $\mathrm{age}(v)$ we can assume that all vertices (except v) are of order one or two and that any hyperelliptic vertex has age zero. If there are other non-trivial edges besides

e, then either they are CPT and we obtain enough age, or similarly as before we obtain $c(\boldsymbol{\tau}) \geq 2/3 + (E_{\mathrm{H}} - 1)/(2E_{\mathrm{H}}) \geq 1$ for $E_{\mathrm{H}} \geq 3$. Otherwise it reduces to EDB, which implies that $c(\boldsymbol{\tau}) \geq \mathrm{age}(v) + 4/\operatorname{lcm}(1, 6) = 1$.

Case that a vertex is of order three. Finally we assume that all vertices are of order one, two, or three, and there exists at least one vertex v of order three. If v admits three cyclicly permuted edges, then it must be paired with another vertex v' of order three, which includes Cases (2), (6), (7), (R1) and (R2). The only combination for $\mathrm{age}(v) + \mathrm{age}(v') < 1$ is where v and v' are of type (2), (R1) or (R2), and then $\mathrm{age}(v) + \mathrm{age}(v') = 2/3$. We can assume that any hyperelliptic vertex has age zero and that there are no CPT edges. Either the three edges do not cross the level passage H, and then $c(\boldsymbol{\tau}) \geq 2/3 + (1/6 + 1/6 + (E_{\mathrm{H}} - 1)/2)/E_{\mathrm{H}} \geq 1$, or we obtain that $c(\boldsymbol{\tau}) \geq 2/3 + (1/3 + 1/3 + 1/3 + (E_{\mathrm{H}} - 3)/2)/E_{\mathrm{H}} \geq 1$ since $E_{\mathrm{H}} \geq 3$ in this case.

Now suppose all vertices of order three have fixed edges only, which includes Cases (3), (4), (5), (R3), (R4) and (R5). For Cases (3), (R3) and (R4), $\mathrm{age}(v) = 2/3$, and they can be treated similarly as in the preceding case. For Case (4), the vertex v admits a unique edge and we can use the same argument as in Case (13).

For Case (5), if v admits one leg and one edge, then the argument is still the same as Case (13). Next suppose v admits two edges e_1 and e_2, going up and down respectively to v_1 and v_2. Again we only need to consider the case that all other vertices (except v) are of order one or two and any hyperelliptic vertex has age zero. It follows that $c(\boldsymbol{\tau}) \geq 1/3 + (1/6 + 1/6 + (E_{\mathrm{H}} - 1)/2)/E_{\mathrm{H}}$. If there is an edge crossing H constrained by GRC, then the corresponding two-level graph has $v_{\mathrm{H}}^{\top} > 1$, and so the $v^{\top}$-contribution (33) gives an additional $1/2$, which is enough. Hence, as in the case of vertices of order two, the edges crossing the level H and the handles of the hyperelliptic trees crossing H, give a contribution of at least $(E_{\mathrm{H}} - 1)/E_{\mathrm{H}} + 1/6$, where we subtract one since the contributing edge might be e_1 or e_2. If $E_{\mathrm{H}} > 1$, then the preceding estimate together with the vertex age is enough. Consider finally the case that $E_{\mathrm{H}} = 1$. Then the graph (modulo trivial level passages) reduces to a tree where both e_1 and e_2 are CPT. Moreover if v_2 is hyperelliptic, then e_2 cannot be RBT by stability of v_2. Therefore, the contribution of e_i to $c(\boldsymbol{\tau})$ is at least $1/3$, which is from either $2c_{e_i} \geq 2/\operatorname{lcm}(2, 3) = 1/3$ or $c_{e_i} \geq 1/\operatorname{lcm}(1, 3) = 1/3$. It follows that $c(\boldsymbol{\tau}) \geq \mathrm{age}(v) + 1/3 + 1/3 = 1$.

For Case (R5) a similar argument as the previous one works. $\square$

A slight modification of the previous proof can be used to show Proposition 5.11.

Proof of Proposition 5.11. Since $1/E_{\Gamma} \leq 1/2$ in the presence of NCT edges, the same proof will work for Proposition 5.11 if we can show that the $v^{\top}$-contribution (33) is not needed if we change the NCT coefficient to $1/2$ as in Proposition 5.11. Indeed in the proof of Proposition 5.12, the $v^{\top}$-contribution is only used in two instances, one in the case of vertices of order one or two and the other in the case of vertices of order three, to ensure that if there are edges crossing H constrained by GRC, then we obtain a contribution of at least $1/2$. In both cases since each edge crossing H constrained by GRC gives a contribution of at least $(1/2) \cdot (1/2) = 1/4$ and there are at least two such edges, these edges contribute at least $1/2$, hence giving enough without using the $v^{\top}$-contribution. $\square$

In order to prove Theorem 1.4 for low genera (the general type result for the minimal odd strata with $g \leq 43$), we need to further reduce the (ℓ/p_e)-coefficients in R^{Γ}_{NC} for certain graphs as follows.

We say that a two-level graph for the minimal stratum in genus g is a *rational multi-banana (RMB)* if it has two vertices only, one on top and one on the bottom, joined by E edges for $E \geq 2$, where the bottom vertex is of genus zero.

Proposition 5.13. *For the minimal strata $\mu = (2g-2)$, we can refine the R^{Γ}_{NC} coefficients of Proposition 5.12 by setting $R^{\Gamma}_{\mathrm{NC}} = 1/\ell_\Gamma$ for RMB graphs with prongs of type $(1^{E_\Gamma - 1}, p)$ where $p \geq 2E_\Gamma - 3$ or with prongs of type $(1^{E_\Gamma-2}, 2, p)$ where $p \geq 2E_\Gamma - 2$, and by setting $R^{\Gamma}_{\mathrm{NC}} = P_{-1}/(E_\Gamma + 1)$ for RMB graphs with at least one prong of order one and at least one prong of order greater than seven, where $P_{-1} = \sum_{e \in E(\Gamma)} 1/p_e$.*

Proof. We will use the same notation as in the proof of Proposition 5.12.

Case that there exists an RMB with prongs $(1^{E_{\mathrm{R}}-1}, p)$ ***or*** $(1^{E_{\mathrm{R}}-2}, 2, p)$**.** Consider first the case in a minimal stratum that Π has a level passage R whose undegeneration gives an RMB graph of prong type $(1^{E_{\mathrm{R}}-1}, p)$ with $p \geq 2$. Let e_p be the edge of prong p. Let $v_i^{\pm}$ be the upper and lower ends of the edges of prong order one and $u^{\pm}$ the upper and lower ends of e_p, where some of the vertices can coincide. We make some observations first. We can assume that R is non-trivial, since otherwise we can apply the initial argument. Since in the minimal strata Π has no local minima other than the unique bottom level vertex, any vertex v_i^+ lower than u^+ admits a unique polar edge of prong one. It follows that p and 1 are the only prong values for the level passages between R and u^+. On the other hand, the subgraph below R is a tree with rational vertices where each vertex goes down to the bottom vertex via a unique path. In particular, other than p, the maximum prong value between R and u^- can be at most $2E_{\mathrm{R}} - 3$, which is less than or equal to p by assumption. Moreover the prong values between R and u^- are all greater than two. It follows that $P_{-1,j}/E_j \geq 1/p$ for $j \in [e_p]$ and any level passage in $[e_p]$ cannot be RMB with at least one prong of order one (except for the prong type $(1^{E-1}, p)$ where $1/\ell_{\mathrm{R}} = 1/p$ is the desired no-compensation coefficient). Since e_p crosses the non-trivial level passage R, we thus conclude from Lemma 5.9 and Lemma 5.10 that

$$c(\boldsymbol{\tau}) \geq \sum_{j \in [e_p]} \frac{\ell_j P_{-1,j}}{E_j} s_j \geq c_{e_p} \geq 1/\operatorname{lcm}(k^+, k^-)$$

where $k^{\pm}$ are the vertex orders of $u^{\pm}$ under $\boldsymbol{\tau}$.

If $u^{\pm}$ are both of order one, then $c_{e_p} \geq 1$ and we are done. If one of them is hyperelliptic, then $c_{e_p} \geq 1/2$, and we can assume that any hyperelliptic vertex is of age zero and that e_p is not constrained by GRC, since otherwise the $v^{\top}$-contribution (33) would give another $1/2$. First suppose u^- is hyperelliptic. Since it has age zero and e_p is not constrained by GRC, the vertex u^- admits a unique polar edge (which is e_p) and a unique zero edge (or leg), which contradicts stability as u^- is of genus zero. Next suppose that u^+ is hyperelliptic and let e_q be a polar edge of maximal prong q (among all the polar edges of u^+). Using as before that there is no local minima other than the bottom level vertex, we conclude that all prong values in $[e_q]$ are q and 1. Therefore, if e_q is not trivial, we obtain from these level passages that $c_{e_q} \geq 1/2$ and hence $c(\boldsymbol{\tau}) \geq c_{e_p} + c_{e_q} \geq 1$. If e_q is trivial, then its upper vertex u_1^+

is hyperelliptic (of age zero), and all other vertices of order one connected to u^+ are on higher level than u_1^+. Hence we can iterate the same argument we used for u_+ to u_1^+, and continue this procedure until we reach a non-trivial edge. The procedure has to stop since the top level of Π cannot have hyperelliptic vertices of age zero (since those have to be of genus zero).

The argument for an RMB graph of prong type $(1^{E_{\mathrm{R}}-2}, 2, p)$ with $p \geq 2$ is similar. Indeed similarly as before, p, 1 and 2 are the only prong values for the level passages between R and u^+. Moreover, arguing as before, the maximum prong value between R and u^- can be at most $2E_{\mathrm{R}} - 2$, which is less than or equal to p by assumption. One additional observation is that in this case $-2(E_{\mathrm{R}} - 2) - 3 - (p+1) + 2g - 2 = -2$, hence p is even. In particular $1/\ell_{\mathrm{R}} = 1/p$ as before. So the previous case of having only order one vertices and the previous case where u^- is hyperelliptic are clear. If u^+ is hyperelliptic of age zero, then under the above notation assume that e_q is non-trivial. Then e_q is the only polar edge and q is even, since $-q-1+p-1 = -2$ and p is even. Indeed, if there are polar edges constrained by GRC, there is a non-trivial level passage with $v^\top > 1$, and hence the $v^\top$-contribution (33) gives another $1/2$. If there are a pair of permuted edges with prong q, we have $2(-q-1)+p-1 = -2$, which is impossible since p is even. It follows that $q \geq 2$ and q is still the largest prong for the level passages in $[e_q]$ (since there might be edges with prongs of order 2 crossing a level passage in $[e_q]$, we had to rule out the case of $q = 1$ in order for the previous sentence to be true). If e_q is trivial, as before we can iterate the procedure until we find a non-trivial edge giving a contribution of $1/2$.

Consider now RMB with prongs of type $(1^{E_{\mathrm{R}}-1}, p)$ or $(1^{E_{\mathrm{R}}-2}, 2, p)$ in the situation of having higher order vertices from Figure 4 and Figure 5.

First we treat the cases in Figure 5 with GRC of order four. If there is such a vertex, then Π has a non-trivial level passage whose corresponding divisor has $v^\top > 1$. Then in this case the $v^\top$-contribution (33) is at least $1/\operatorname{lcm}(k_1, k_2)$, where k_i are the orders of the vertices joined by the special edge with the property that all the level passages crossed by it have $v^\top > 1$. It is easy to check that this contribution, together with the age contribution from the vertices, is enough.

Next consider the cases in Figure 4. Since the Cases (4), (9) and (13) give CPT edges, which cannot cross R, we do not need to consider these cases. Indeed we have seen that, using the above notation, it is enough to consider the contribution from the edges e_p and e_q. Since e_p cannot be a CPT edge, the only possibility would be that e_q is the CPT edge, but in this case $c_{e_q} \geq 2/6$, which, together with c_{e_p} and the age contribution from the vertices, is enough. Moreover, Case (8) needs to join another vertex of order four, thus giving enough vertex age. The remaining cases of order four are (10) and (11). Since these cases correspond to vertices of positive genus, the vertex u^- cannot be of this type. If u^+ is not of type (10) or (11), then we obtain enough age from $c_{e_p} \geq 1/2$ and the vertex age. If u^+ is of type (10) or (11), we obtain that $c_{e_p} + c_{e_q} + \operatorname{age}(u^+) \geq 1/4 + 1/4 + 1/2 = 1$.

We are left with considering the cases of order three. The only cases of genus zero are (R1) and (R2), both having three permuted edges which need to join another vertex of order three. Then the age contribution of at least $2/3$ from these two vertices, together with the additional $v^\top$-contribution (33) of at least $1/6$ and the edge contribution of at least $1/6$, is enough. The remaining cases are all of positive genus, hence they cannot correspond to the vertex u^-. If u^- is hyperelliptic, by the argument above we have $\mathrm{age}(u^-) \geq 1/2$. Then the vertex age is at least $1/2 + 1/3$ and $c_{e_p} \geq 1/6$, hence we obtain enough age. Consider finally the case where u^- is of order one, and then $c_{e_p} \geq 1/3$. If the vertex age is at least $2/3$, we obtain enough age. Hence we are left to consider the case where only one vertex of order three is present and it has age $1/3$, which means Cases (5) and (R5) (Case (4) admits a non-trivial CPT edge which gives enough contribution). If the upper end of e_q is of order one, then we obtain $c_{e_p} + c_{e_q} \geq 2/3$, which together with the vertex age, is enough. If the upper end of e_q is hyperelliptic (necessarily of age zero), then by the same analysis for u^+ we obtain another edge giving a contribution of at least $1/2$, hence we obtain enough together with the contribution of $c_{e_p} + c_{e_q} \geq 1/3 + 1/6$ and the vertex age.

Case that there exists an RMB with a prong of order one and a prong of large order. Consider a graph Π such that one of its undegenerations is an RMB graph with at least one prong of order one and one prong of order greater than seven. We can also assume that there is no special RMB level of prong type $(1^{E_{\mathrm{R}}-1}, p)$ or $(1^{E_{\mathrm{R}}-2}, 2, p)$. As before, denote by H the level passage with the largest number of edges crossing and by R the RMB level passage, which we can assume to be non-trivial.

Consider first the case of having only vertices of order one. Since any non-trivial edge in Π yields a contribution of at least $1/(E_{\mathrm{H}} + 1)$, the E_{H} edges crossing H give a contribution of at least $E_{\mathrm{H}}/(E_{\mathrm{H}} + 1)$. If there is a non-trivial level passage different from H, then we gain the additional contribution to reach age one. If H is the only non-trivial level passage, then by Lemma 5.9 and Lemma 5.10 we obtain $s_{\mathrm{H}}\ell_{\mathrm{H}}/p_e \in \mathbb{N}$ for all edges e crossing H, which is impossible for $0 < s_{\mathrm{H}} < 1$.

Consider now the case where the vertices have order one or two. Since the edge contribution is at least $E_{\mathrm{H}}/2(E_{\mathrm{H}}+1)$, if there is a contribution of $1/2$ from a vertex of order two or from a non-trivial level passage with $v^\top > 1$, we can argue similarly as before. Indeed any edge yields a contribution of at least $1/2(E_{\mathrm{H}} + 1)$. Hence if there is a non-trivial level passage different from H, we obtain enough. As before we can also rule out the case that H is the only non-trivial level passage. Therefore, it suffices to consider the case where all vertices of order two are hyperelliptic of age zero and all non-trivial level passages have $v^\top = 1$.

Since we can assume that $v_{\mathrm{H}}^\top = 1$, using the same analysis as in the general case, i.e., the case with vertices of order two in the proof of Proposition 5.11, we obtain a contribution of at least $c(\boldsymbol{\tau}) \geq E_{\mathrm{H}}/(E_{\mathrm{H}} + 1)$ by considering the contributions of the E_{H} edges together with the contributions of the handles of hyperelliptic trees crossing H. We thus need to find an additional contribution of at least $1/(E_{\mathrm{H}} + 1)$.

Let e be the special edge with prong one and let $v^\pm$ be the vertices at its upper and lower ends. If e is a handle of type two, i.e., it is a fixed edge and at least one of its endpoints is hyperelliptic, then v^+ cannot be hyperelliptic of age zero and hence v^-, being hyperelliptic of age zero, has a pair of permuted zero edges of prong one. This is impossible since the graph below v^- is a tree with rational vertices by the RMB assumption for the level R. Hence if e does not cross H, we obtain the additional contribution we need. If e crosses H and is of type three, i.e., it belongs to a pair of permuted edges with prong one, then v^+ admits a single polar edge e_1, which is a handle of prong one. Moreover, v^- also admits a single zero edge e_2 of prong three, which (modulo trivial level passages) is a handle since the graph below v^- is a tree. It follows that v^+ (resp. v^-) is (modulo trivial level passages) on the top (resp. bottom) level of Π (since otherwise we would gain an additional contribution from edges joining trivial vertices or hyperelliptic trees not crossing H). Moreover, they are the only vertices on the top and bottom levels, since $v_{\mathrm{H}}^{\top} = 1$ and Π has a unique bottom level. Let e_p be the special edge crossing R with prong $p \geq 8$. If e_p crosses only the level passages crossed by e or the upper handle e_1 (which have both prong order one), then $c_e + c_{e_1} \geq pc_{e_p} \geq p/2 \geq 2$. Since the original calculation only used a contribution of 1 for $c_e + c_{e_1}$, we have found the desired missing contribution. Therefore, we can assume that e_p joins the top and bottom vertices (since otherwise we would get enough contribution from additional edges joining trivial vertices or hyperelliptic trees). In this case e_p joins two trivial vertices and since $p \geq 8$, then $c_e + c_{e_1} + c_{e_2} \geq pc_{e_p}/3 \geq p/3 \geq 3/2 + 1$. This is enough since in the original computation the contribution given by $c_e + c_{e_1} + c_{e_2}$ was $3/2$.

Now we have reduced to the situation where e crosses H and it is of type one, i.e., the vertices $v^{\pm}$ are trivial. In the following arguments we can assume that edges joining trivial vertices are non-trivial (e.g., by collapsing those trivial edges and merging the corresponding trivial vertices). If v^+ admits a polar edge going up to a hyperelliptic vertex of age zero, then the hyperelliptic tree associated to this vertex does not cross H and hence yields an additional contribution of at least $1/(E_{\mathrm{H}} + 1)$ given by its (at least) two handles. If v^+ admits a polar edge going up to another trivial vertex, then we gain the desired extra $1/(E_{\mathrm{H}} + 1)$ from this extra edge, which is neither a handle nor crossing H. From now on we can assume that v^+ admits zero edges only. If there is an edge crossing H not joining v^+, since $v_{\mathrm{H}}^{\top} = 1$ and since v^+ is above H (as e crosses H), then there is a path completely above H joining the upper vertex of this edge and v^+. Then this path gives enough contribution either because there is one edge joining two trivial vertices or because it is a hyperelliptic tree. Hence we can assume that v^+ is the common upper vertex for all edges crossing H. Suppose v^- is the lowest vertex reached by edges crossing R. In this case, since by assumption there is an edge e_p with prong $p \geq 8$ crossing R, using that e has prong one we obtain $c_e \geq pc_{e_p} \geq 8(1/2) \geq 2$. Since in the original estimate we only used $c_e \geq 1$, we gain the extra contribution we needed. We are left to show that if there is an edge crossing R, then its ending vertex is above v^- or we get enough compensation. If v^- is on the bottom level of Π, then we are done. If not, v^- joins a lower level via an edge e'. If the lower end of e' is trivial, we obtain the desired extra contribution of at least $1/(E_{\mathrm{H}} + 1)$. If the lower end of e' is hyperelliptic (which cannot be the lowest level of Π), then the hyperelliptic tree starting with handle e' gives enough contribution.

Consider finally the situation where higher order vertices appear. If $v_{\mathrm{H}}^{\top} > 1$, we obtain a contribution of at least $(E_{\mathrm{H}}/(E_{\mathrm{H}} + 1) + 1)/\max(\mathrm{lcm}(k_1, k_2))$, where k_i run among the order of the vertices. First, the case of H being the only non-trivial level passage is impossible since R is non-trivial with $v_{\mathrm{R}}^{\top} = 1$. If now there is at least another non-trivial level passage, we have an additional edge contribution of $1/(E_{\mathrm{H}} + 1)/\max(\mathrm{lcm}(k_1, k_2))$. By inspecting Figure 5 and considering also the vertex age contribution, the only possibility is to have at most one higher order vertex with age $1/3$, which means Cases (5) and (R5) (Cases (4) and (13) can be excluded since a non-trivial CPT edge yields enough contribution). In this case the $v^{\top}$-contribution is at least $1/6$. Let v_1 and v_2 be the two vertices joined to the special vertex of order three. If v_1, v_2 are both trivial and there are in total at least $E_{\mathrm{H}} + 2$ non-trivial edges, then we have the age estimate $c(\boldsymbol{\tau}) \geq 1/3 + 1/6 + (1/3 + 1/3 + E_{\mathrm{H}}/2)/(E_{\mathrm{H}} + 1) > 1$. If there are exactly $E_{\mathrm{H}} + 1$ non-trivial non-CPT edges, then $v_{\mathrm{H}}^{\top} = 1$. Suppose one of the v_i is hyperelliptic and the other is trivial. Then the hyperelliptic vertex needs to go up or down further (to avoid CPT and since the bottom minimal rational vertex cannot be hyperelliptic). Hence there are in total at least $E_{\mathrm{H}} + 2$ non-trivial edges. Then we have $c(\boldsymbol{\tau}) \geq 1/3 + 1/6 + (1/3 + 1/6 + E_{\mathrm{H}}/2)/(E_{\mathrm{H}} + 1) = 1$. Suppose finally that v_1 and v_2 are both hyperelliptic. Then they need to go up and down respectively for the same reason. Hence there are in total at least $E_{\mathrm{H}} + 3$ non-trivial edges, and we have $c(\boldsymbol{\tau}) \geq 1/3 + 1/6 + (1/6 + 1/6 + (E_{\mathrm{H}} + 1)/2)/(E_{\mathrm{H}} + 1) > 1$.

At this point we have reduced to the situation of $v_{\mathrm{H}}^{\top} = 1$ for any level passages, and we can argue similarly as in the non-special RMB situation, and use the same notation as in the case of vertices of order two in the proof of Proposition 5.11, where we considered three types of edges. Besides these three types, there can be now E_4 edges of a fourth type joining at least one vertex of order greater than two. If $E_4 = 0$ and the handles of the hyperelliptic trees crossing H do not join higher order vertices, then as before the level passage H gives a contribution of at least $E_{\mathrm{H}}/(E_{\mathrm{H}} + 1) \geq 2/3$ for $E_{\mathrm{H}} \geq 2$, which is enough together with the additional age from the higher order vertices. One can argue similarly if there are some handles that join higher order vertices. If $E_{\mathrm{H}} = 1$, then we have only CPT edges, and we have already discussed this case. If $E_4 > 0$, then we can only have $E_4 = 1, 2, 3$. If $E_4 = 3$, then there are at least two vertices of order three and the vertex age is at least $2/3$, which together with $(E_{\mathrm{H}} - E_4 + 3/3)/(E_{\mathrm{H}} + 1) \geq 1/3$ for $E_{\mathrm{H}} \geq 4$ is enough. If $E_{\mathrm{H}} = E_4 = 3$, then H cannot be R, since R is crossed by at least two edges with different prongs. If $E_4 = 2$, one can argue similarly. If $E_4 = 1$, then we get a contribution from H of at least $(E_{\mathrm{H}} - E_4)/(E_{\mathrm{H}} + 1)$, which is at least $1/3$ for $E_{\mathrm{H}} \geq 2$ and at least $2/3$ for $E_{\mathrm{H}} \geq 5$. Hence if $E_{\mathrm{H}} \geq 5$ or $E_{\mathrm{H}} = 1$ we get enough contribution. If $2 \leq E_{\mathrm{H}} \leq 4$, the only two possibilities left are Cases (5) and (10) (as before Cases yielding CPT edges do not need to be considered). For Case (10), which gives a vertex age of $1/2$, we obtain a refined estimate $(E_{\mathrm{H}} - 1 + 1/4)/(E_{\mathrm{H}} + 1) + 1/E_{\mathrm{H}} \geq 1/2$ for $E_{\mathrm{H}} \geq 2$, where the last $1/E_{\mathrm{H}}$ term comes from the additional edge attached to the special vertex. Similarly for Case (5), which gives a vertex age of $1/3$, we obtain $(E_{\mathrm{H}} - 1 + 1/6)/(E_{\mathrm{H}} + 1) + 1/E_{\mathrm{H}} \geq 2/3$ for $E_{\mathrm{H}} \geq 2$. □

Remark 5.14. We give some examples to illustrate that the choices of the coefficients in R_{NC}^{Γ} are delicate.

Take three vertices, each on a different level, where the middle vertex joins the top and the bottom each by a single edge of prong p_1 and p_2, respectively, the top joins the bottom by $E - 1$ edges of prong all equal to $p_3 = p_1 + p_2$, and p_1, p_2 are relatively prime. Suppose $\boldsymbol{\tau}$ acts trivially on the vertices and acts on the level parameters t_1 and t_2 by $t_i \mapsto e^{2\pi i/p} t_i$, i.e., the arguments of the action (mod $2\pi i$) are $s_1 = s_2 = 1/p_3$ as in the proof of Lemma 5.9. In this case $\ell_1 = p_1 p_3$, $\ell_2 = p_2 p_3$, and they satisfy the requirements that $(\ell_1/p_1)s_1 = 1$, $(\ell_1/p_2)s_2 = 1$, and $(\ell_1/p_3)s_1 + (\ell_2/p_3)s_2 = 1$. We have $\ell_1(1/p_1 + (E-1)/p_3)s_1 + \ell_2(1/p_2 + (E-1)p_3)s_2 = E + 1$, hence we cannot use a coefficient smaller than $1/(E+1)$ (and indeed we use $1/E$ for general NCT edge types in Proposition 5.12).

For the RBT coefficient, since $\ell = p$ for the unique RBT edge and $\ell/p - 1 = 0$, i.e., we do not make any compensation, it is clearly sharp.

For the general CPT coefficient (i.e., OCT), take Case (5) of order 3 and age $1/3$ as the middle vertex sitting in between two hyperelliptic trees of age zero. Let $s_1 = s_2 = 1/6 = 1/\operatorname{lcm}(2,3)$ for the action on the level parameters of the two handles. Then $c(\boldsymbol{\tau}) = 1/3 + 2 \cdot (1/6 + 1/6) = 1$, which implies that we cannot reduce the OCT coefficient to be smaller than 2. Note that in this case the genus of the top or the bottom can be almost arbitrary after a divisorial undegeneration (except that it cannot be zero since Case (5) of genus one is contained in a middle level, which singles out RBT).

For the EDB coefficient, take Case (5) as a top or a bottom vertex joining a hyperelliptic tree of age zero, and let $s = 1/6$ for the action on the level parameter of the handle. Then $c(\boldsymbol{\tau}) = 1/3 + 4 \cdot (1/6) = 1$, which shows that the EDB coefficient 4 is necessary.

For the minimal strata RMB coefficients in Proposition 5.13, consider a triangle graph with one top vertex v_1, one middle vertex v_2, and one bottom vertex v_3, where v_2 is hyperelliptic of age zero, v_1 and v_3 are trivial, and moreover v_3 is rational with the unique marked zero. Let p_1 be the higher short edge, p_2 the lower short edge, and p_3 the long edge satisfying that $2p_3 = p_1 + p_2$. Suppose p_1, p_2, p_3 are odd and pairwise relatively prime. Then $\ell_1 = p_1 p_3$ and $\ell_2 = p_2 p_3$. The top level undegeneration is a non-RMB banana graph (as v_2 cannot be rational with only two edges and no legs) and the bottom level undegeneration is an RMB. Take $s_1 = s_2 = 1/(2p_3)$ which satisfies the half-integer requirement along the short edges and the integer requirement along the long edge. Suppose we use the coefficient $1/2$ for the top non-RMB banana graph and the coefficient $1/(E_{\mathrm{R}} + 1) = 1/3$ for the bottom RMB. Then we obtain $c(\boldsymbol{\tau}) = (1/2) \cdot \ell_1 s_1 (1/p_1 + 1/p_3) + (1/3) \cdot \ell_2 s_2 (1/p_2 + 1/p_3) = 3/4 + p_1/(12p_3)$ which can be smaller than 1. However in this case $p_2 \neq 1$ since v_2 cannot be rational, which also implies that $p_3 \neq 1$ by the relation $2p_3 = p_1 + p_2$. Therefore, imposing a prong of order one when reducing the coefficients of RMB graphs in the minimal strata makes sense.

Finally we show that an extra contribution from multiple top vertices (i.e., the $v^{\top}$-term) is necessary in Proposition 5.12. Consider a three-level graph where $u_1, \ldots, u_n$ are trivial vertices on the top level, v is a hyperelliptic vertex (of age zero) on the middle level, and w is a trivial vertex on the bottom level. Suppose each u_i joins v by a short edge of prong p_1, v joins w by a short edge of prong p_2, and each u_i joins w by a long edge of prong p_3, where $p_1 + p_2 = 2p_3$ and p_1, p_2, p_3 are pairwise relatively prime. The GRC imposes zero residue for each polar edge of v, hence the age of v can be zero. Moreover, every edge in every undegeneration of this graph is NCT. Take $s_1 = s_2 = 1/(2p_3)$ which satisfies the half-integer requirement along every short edge and the integer requirement along every long edge. The top level passage has $2n$ edges and the bottom level passage has $n+1$ edges. Then we obtain $c(\boldsymbol{\tau}) = 1/(2n) \cdot \ell_1 s_1 (n/p_1 + n/p_3) + 1/(n+1) \cdot \ell_2 s_2 (1/p_2 + n/p_3) \sim 3/4 < 1$ for $n \gg 0$ and $p_3 \gg p_2$. Hence in this case an extra contribution from $v^{\top}$ is needed.

6. Pullback classes and the canonical class

In this section we recall basic properties of the tautological ring of the moduli space of multi-scale differentials and express some divisor classes in terms of standard generators. We apply this to the formula for the canonical bundle and also compute the classes of divisors of Brill–Noether type that are pulled back from the moduli space of curves.

The divisorial part of the tautological ring is generated by the ψ-classes and boundary classes. It contains standard tautological divisor classes ξ, λ_1, κ_1, which are all proportional in the strata interior but can differ along the boundary (see e.g., [Che19]). For their conversion we define rational numbers

$$\kappa_\mu \;=\; \sum_{m_i \neq -1} \frac{m_i(m_i+2)}{m_i+1} \;=\; 2g-2+s+\sum_{m_i\neq -1}\frac{m_i}{m_i+1} \tag{34}$$

for any signature $\mu=(m_1,\dots,m_n)$, where s is the number of entries equal to -1 (i.e. the number of simple poles). The values $\kappa^\perp := \kappa_{\mu_\Gamma^\perp}$ and $\kappa^\top := \kappa_{\mu_\Gamma^\top}$ are similarly defined for the bottom and top level strata of Γ, *including the edges as legs*. In particular

$$\kappa^\perp + \kappa^\top \;=\; \kappa_\mu\,.$$

This constant κ_μ previously appeared (with an additional factor $\frac{1}{12}$) in [EKZ14] for relating sums of Lyapunov exponents and (area) Siegel-Veech constants in the strata of holomorphic differentials. Our definition here includes meromorphic signatures as well. We also define the ψ-class over simple poles to be

$$\psi_{-1} \;=\; \sum_{m_i=-1}\psi_i\,.$$

The main conversion result of divisor classes we need is the following relation, whose proof is given in Section 6.1.

Proposition 6.1. *In the tautological ring we have the relation*

$$\kappa_\mu\xi \;=\; \psi_{-1}+12\lambda_1-[D_h]-\sum_{\Gamma\in\mathrm{LG}_1}\ell_\Gamma\kappa_{\mu_\Gamma^\perp}[D_\Gamma]\,. \tag{35}$$

The canonical class of the coarse moduli space follows from the canonical class of the stack computed in [CMZ20b], using the above conversion formula and taking into account the branching behavior at the boundary. Recall from Section 2.3 the definition of graphs Γ of type HTB, HBT and HBB (and their prime versions for meromorphic strata) as well as the corresponding ramification divisors D_Γ^{H}. Recall also that N denotes the dimension of the (unprojectivized) strata.

Proposition 6.2. *Let μ be a holomorphic signature not of type $(2m, 2g-2-2m)$ (and the hyperelliptic component is excluded if $\mu=(2g-2)$), or a meromorphic signature not of type $(2m_1, 2m_2, 2g-2-2m_1-2m_2)$ (and the hyperelliptic component is excluded if $\mu=(2g-2+2m,-2m)$). Then the class of the canonical bundle of the coarse moduli space $\mathbb{P}\mathrm{MS}(\mu)$ is given by*

$$\begin{aligned}\frac{\kappa_\mu}{N}\,\mathrm{c}_1\big(K_{\mathbb{P}\mathrm{MS}(\mu)}\big) \;&=\; \psi_{-1}+12\lambda_1-\Big(1+\frac{\kappa_\mu}{N}\Big)[D_h]\\ &-\sum_{\Gamma\in\mathrm{LG}_1}\Big(\ell_\Gamma\kappa_{\mu_\Gamma^\perp}-\frac{\kappa_\mu}{N}(\ell_\Gamma N_\Gamma^\perp-1)\Big)[D_\Gamma]-\frac{\kappa_\mu}{N}\sum_{\substack{\Gamma\ is\ \mathrm{HTB}\ or\\ \mathrm{HBT}\ or\ \mathrm{HBB}}}[D_\Gamma^{\mathrm{H}}]\end{aligned} \tag{36}$$

in $\mathrm{CH}^1(\mathbb{P}\mathrm{MS}(\mu))=\mathrm{CH}^1(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu))$, where for meromorphic signatures the last sum is for graphs of type HTB', HBT' and HBB'.

We remark that for μ of holomorphic type $(2m, 2g-2-2m)$ or of meromorphic type $(2m_1, 2m_2, 2g-2-2m_1-2m_2)$ the above expressions have to be modified by the class of the ramification divisor in the interior described by Proposition 2.2.

The proof of Proposition 6.2 together with the variants for connected components of the strata is given in Section 6.2.

6.1. **Relations among tautological classes.** We denote by

$$R^\bullet(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)) \subset \mathrm{CH}^\bullet(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)) \tag{37}$$

the *tautological ring*, being the smallest subring that contains the ψ-classes, that is closed under the push-forward of level-wise clutching and forgetting a marked point, and moreover that contains the class of the horizontal divisor D_h (including the class of each component if D_h is reducible).[4] Note that D_h is irreducible in each holomorphic stratum (component), but in general it can be reducible for the meromorphic strata.

The tautological ring contains all boundary strata classes and standard tautological classes such as the κ-classes. Let $\pi\colon \mathcal{X} \to \overline{B}$ be the universal family with $s_i\colon \overline{B} \to \mathcal{X}$ the universal sections, $S_i \subset \mathcal{X}$ their images and ω_π the relative cotangent bundle. Define the Miller–Morita–Mumford class $\kappa_1 = \pi_*(\mathrm{c}_1(\omega_\pi)^2)$. We give a closed expression for κ_1 as follows.

Proposition 6.3. *The class κ_1 can be expressed in terms of the standard generators of $R^\bullet(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu))$ as*

$$\kappa_1 \;=\; -\psi_{-1} + \kappa_\mu\xi + \sum_{\Gamma\in\mathrm{LG}_1} \ell_\Gamma\Big(\kappa_{\mu_\Gamma^\perp} - \sum_{e\in E(\Gamma)} \frac{1}{p_e}\Big)[D_\Gamma]$$

where ψ_{-1} is the sum of ψ-classes associated to marked simple poles.

We remark that the above expression can be converted to the Arbarello–Cornalba κ-class via the relation

$$\kappa_1^{\mathrm{AC}} \;=\; \pi_*\Big(\mathrm{c}_1\Big(\omega_\pi\Big(\sum_{i=1}^n S_i\Big)\Big)^2\Big) \;=\; \kappa_1 + \psi$$

where $\psi = \sum_{i=1}^n \psi_i$ is the total ψ-class.

Proof. The prescribed vanishing of the differentials in the universal family along the sections S_i implies that

$$\mathrm{c}_1(\omega_\pi) \;=\; \pi^*\xi + \sum_{i=1}^n m_i S_i + \sum_{\Gamma\in\mathrm{LG}_1} \ell_\Gamma[\mathcal{X}_\Gamma^\perp]$$

where $\mathcal{X}_\Gamma^\perp$ is the vertical vanishing divisor over the locus with level graph Γ. We compute

$$\begin{aligned}
\kappa_1 \;&=\; \pi_*\Big(\Big(\pi^*\xi + \sum_{i=1}^n m_i S_i + \sum_{\Gamma\in\mathrm{LG}_1} \ell_\Gamma[\mathcal{X}_\Gamma^\perp]\Big)^2\Big) \\
&=\; -\sum_{i=1}^n m_i^2\psi_i + \pi_*\Big(\Big(\sum_{\Gamma\in\mathrm{LG}_1} \ell_\Gamma[\mathcal{X}_\Gamma^\perp]\Big)^2\Big) + (4g-4)\xi + 2\sum_{i=1}^n \sum_{\Gamma\in_i\mathrm{LG}_1} m_i\ell_\Gamma[D_\Gamma] \\
&=\; (4g-4)\xi - \sum_{i=1}^n m_i^2\psi_i + 2\sum_{i=1}^n \sum_{\Gamma\in_i\mathrm{LG}_1} m_i\ell_\Gamma[D_\Gamma] + \sum_{\Gamma\in\mathrm{LG}_1} \ell_\Gamma^2\pi_*([\mathcal{X}_\Gamma^\perp]^2)
\end{aligned}$$

where ${}_i\mathrm{LG}_1$ means the i-th marked point is in lower level.

[4] A version of (small) tautological ring without D_h was also considered in [CMZ20b] for the purpose of running `diffstrata`.

Next we evaluate $\pi_*([\mathcal{X}_\Gamma^\perp]^2)$. Since $[\mathcal{X}_\Gamma^\perp]+[\mathcal{X}_\Gamma^\top]=\pi^*[D_\Gamma]$, it suffices to evaluate $\pi_*([\mathcal{X}_\Gamma^\perp]\cdot[\mathcal{X}_\Gamma^\top])$, which is a class supported on D_Γ with suitable multiplicity. Geometrically $\mathcal{X}_\Gamma^\perp$ and $\mathcal{X}_\Gamma^\top$ intersect along the (vertical) edges of Γ. For $e\in E(\Gamma)$, to figure out its contribution to the multiplicity, it suffices to take a general one-parameter family C crossing through D_Γ. The local singularity type of $\mathcal{X}|_C$ at e is ℓ_Γ/p_e, i.e., locally the corresponding node is defined by $xy=t^{\ell_\Gamma/p_e}$ where t is the base parameter. Hence the local contribution is p_e/ℓ_Γ (as the reciprocal of the exponent of t). We conclude that

$$\pi_*([\mathcal{X}_\Gamma^\perp]^2) \;=\; -\pi_*([\mathcal{X}_\Gamma^\perp]\cdot[\mathcal{X}_\Gamma^\top]) \;=\; -\frac{1}{\ell_\Gamma}\Big(\sum_{e\in E(\Gamma)} p_e\Big)[D_\Gamma]\,.$$

Moreover by the relation at the beginning we have

$$\xi \;=\; (m_i+1)\psi_i - \sum_{\Gamma\in_i \mathrm{LG}_1} \ell_\Gamma [D_\Gamma] \tag{38}$$

(see e.g., [CMZ20b, Proposition 8.2]) and consequently

$$(2g-2)\xi \;=\; \sum_{i=1}^n (m_i^2+m_i)\psi_i - \sum_{i=1}^n \sum_{\Gamma\in_i\mathrm{LG}_1} m_i\ell_\Gamma[D_\Gamma]\,.$$

In particular, ψ_i can be converted to ξ with boundary classes as long as $m_i\neq -1$, and $\xi=-\sum_{\Gamma\in_i\mathrm{LG}_1}\ell_\Gamma[D_\Gamma]$ for $m_i=-1$. It follows that

$$\begin{aligned}
\kappa_1+\psi_{-1} \;&=\; (2g-2+s)\xi + \sum_{m_i\neq -1} m_i\psi_i + \sum_{\Gamma\in\mathrm{LG}_1}\ell_\Gamma\Big(\sum_{\substack{i\in L(\Gamma)\\ m_i\neq -1}} m_i - \sum_{e\in E(\Gamma)} p_e\Big)[D_\Gamma] \\
&=\; \kappa_\mu\xi + \sum_{m_i\neq -1}\frac{m_i}{m_i+1}\sum_{\Gamma\in_i\mathrm{LG}_1}\ell_\Gamma[D_\Gamma] + \sum_{\Gamma\in\mathrm{LG}_1}\ell_\Gamma\Big(\sum_{\substack{i\in L(\Gamma)\\ m_i\neq -1}} m_i - \sum_{e\in E(\Gamma)} p_e\Big)[D_\Gamma] \\
&=\; \kappa_\mu\xi + \sum_{\Gamma\in\mathrm{LG}_1}\ell_\Gamma\Big(\sum_{\substack{i\in L(\Gamma)\\ m_i\neq -1}}\frac{m_i^2+2m_i}{m_i+1} - \sum_{e\in E(\Gamma)} p_e\Big)[D_\Gamma] \\
&=\; \kappa_\mu\xi + \sum_{\Gamma\in\mathrm{LG}_1}\ell_\Gamma\Big(\kappa_{\mu_\Gamma^\perp} - \sum_{e\in E(\Gamma)}\frac{1}{p_e}\Big)[D_\Gamma]
\end{aligned}$$

as claimed, where $L(\Gamma)$ denotes the lower level of Γ and in the last step we used that a vertical edge e with pole order $-p_e-1$ contributes $-p_e+1/p_e$ in $\kappa_{\mu_\Gamma^\perp}$. $\square$

Next we compute the first Chern class of the Hodge bundle λ_1 as shown in Proposition 6.1. For that purpose we need to pull back boundary divisor classes from the moduli space of curves to the moduli space of multi-scale differentials. Let $\delta_{[n]}$ be the total boundary divisor class in $\overline{\mathcal{M}}_{g,n}$. Denote by $f_{[n]}\colon \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)\to\overline{\mathcal{M}}_{g,n}$ the natural map remembering the underlying pointed stable curves only.

Lemma 6.4. *The pullback of $\delta_{[n]}$ to the moduli space of multi-scale differentials has divisor class*

$$f_{[n]}^*(\delta_{[n]}) \;=\; [D_h] + \sum_{\Gamma\in\mathrm{LG}_1}\ell_\Gamma\Big(\sum_{e\in E(\Gamma)}\frac{1}{p_e}\Big)[D_\Gamma]\,.$$

Proof. The local equation of the universal curve over $\overline{\mathcal{M}}_{g,n}$ is $xy = t$ for the node defining each boundary divisor. In the family over $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ the local equation is $xy = t$ for D_h by the horizontal plumbing [BCGGM2, Equation (12.6)] and $xy = t^{\ell_\Gamma/p_e}$ for each (vertical) edge e in a graph $\Gamma \in \mathrm{LG}_1$ by Equation (12.8) in loc. cit. We thus obtain the sum on the right-hand side of the desired equation and each summand contributes ℓ_Γ/p_e. □

Proof of Proposition 6.1. Recall the well-known relation $12\lambda_1 = \kappa_1 + \delta_{[n]}$ on $\overline{\mathcal{M}}_{g,n}$ (see e.g. [ACG11, Chapter XIII, Equation (7.7)]). Combining it with Lemma 6.4 and Proposition 6.3 we thus conclude the desired formula. □

For convenience of later calculations we combine Proposition 6.1 and (38) to get

$$\psi_i \;=\; \frac{1}{\kappa_\mu(m_i+1)}\Big(\psi_{-1} + 12\lambda_1 - [D_h] - \sum_{\Gamma\in\mathrm{LG}_1}\big(\kappa_{\mu_\Gamma^\perp} - \delta_{i,\perp}\kappa_\mu\big)\ell_\Gamma[D_\Gamma]\Big) \tag{39}$$

if $m_i \neq -1$, where we define the 'Kronecker' symbol $\delta_{i,\perp}$ to be 1 if the i-th leg is on bottom level and zero otherwise.

6.2. **The canonical bundle formula.** Our goal here is to prove the canonical bundle formula in Proposition 6.2. We continue to use $\mathbf{D}_\Gamma$, $\mathbf{D}_h$, etc, for the reduced boundary divisors of the coarse moduli space $\mathbb{P}\mathrm{MS}(\mu)$, to distinguish from the divisors D_Γ, D_h, etc, in the stack $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. We use $\mathbf{D}$ and D to denote respectively the total boundary divisors. We will express the canonical classes in standard generators in $\mathrm{CH}^1(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu))$, both for the stack and the coarse moduli space, so that even for the latter no boldface objects appear in the final formulas. We start with:

Proposition 6.5. *The class of the log canonical bundle of the smooth Deligne–Mumford stack $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is given by*

$$\frac{\kappa_\mu}{N}\,\mathrm{c}_1\Big(\Omega^d_{\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)}(\log D)\Big) \;=\; \psi_{-1} + 12\lambda_1 - [D_h] + \sum_{\Gamma\in\mathrm{LG}_1}\ell_\Gamma\Big(\kappa_\mu\frac{N_\Gamma^\perp}{N} - \kappa_{\mu_\Gamma^\perp}\Big)[D_\Gamma]$$

in $\mathrm{CH}^1(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu))$, where $d = \dim \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ and $N_\Gamma^\perp$ is the dimension of the unprojectivized bottom level stratum in D_Γ.

Proof. This follows from [CMZ20b, Theorem 1.1] and (35). □

Next we pass to the coarse moduli space, still in the logarithmic context:

Proposition 6.6. *Let μ be a holomorphic signature not of type $(2m, 2g-2-2m)$ (and the hyperelliptic component is excluded if $\mu = (2g-2)$), or a meromorphic signature not of type $(2m_1, 2m_2, 2g-2-2m_1-2m_2)$ (and the hyperelliptic component is excluded if $\mu = (2g-2+2m, -2m)$). Then the class of the reflexive log canonical bundle of the coarse moduli space $\mathbb{P}\mathrm{MS}(\mu)$ is given by*

$$\frac{\kappa_\mu}{N}\,\mathrm{c}_1\Big(\Omega^{[d]}_{\mathbb{P}\mathrm{MS}(\mu)}(\log \mathbf{D})\Big) \;=\; \psi_{-1} + 12\lambda_1 - [D_h] + \sum_{\Gamma\in\mathrm{LG}_1}\ell_\Gamma\Big(\kappa_\mu\frac{N_\Gamma^\perp}{N} - \kappa_{\mu_\Gamma^\perp}\Big)[D_\Gamma]$$

in $\mathrm{CH}^1(\mathbb{P}\mathrm{MS}(\mu)) = \mathrm{CH}^1(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu))$, where $d = \dim \mathbb{P}\mathrm{MS}(\mu)$.

Proof of Proposition 6.6 and Proposition 6.2. By the assumption Proposition 2.2 says that the map $\varphi\colon \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) \to \mathbb{P}\mathrm{MS}(\mu)$ is unramified in the interior. For any linear combination of boundary divisors $\sum a_i D_i$ in the stack, suppose φ is ramified with order e_i at D_i and with image $\mathbf{D}_i$ in the coarse moduli space. Then the ramification formula passing from the stack to the coarse moduli space (e.g. [HH09, Proposition A.13]) gives an equality of $\mathbb{Q}$-Cartier divisors

$$\varphi^*\Big(K_{\mathbb{P}\mathrm{MS}(\mu)} + \sum_i \frac{e_i - 1 + a_i}{e_i}\mathbf{D}_i\Big) \;=\; K_{\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)} + \sum_i a_i D_i\,. \tag{40}$$

Setting all $a_i = 1$, i.e., taking the combination $D_h + \sum_\Gamma D_\Gamma$, Proposition 6.6 thus follows from the above together with Proposition 6.5. Next we can set $a_i = 0$ for unramified boundary divisors and $a_i = -1$ for ramified boundary divisors (of order two). Then Proposition 6.2 follows from Proposition 2.5 and Proposition 2.6. □

Next we explain how to apply the above formulas to each connected component of a disconnected stratum. Recall that the connected components of strata of holomorphic differentials have been classified by Kontsevich–Zorich ([KZ03]). For special signatures these connected components are distinguished by the parity of the spin structure, odd or even, and by consisting entirely of hyperelliptic differentials. We denote these components by an upper index odd, even or hyp respectively. Note that if a holomorphic stratum has three components, then the hyperelliptic component has a fixed spin parity depending on g. We emphasize that the superscript odd or even excludes the hyperelliptic component with that spin parity. The smooth compactification $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ still separates these components and we distinguish them by the same superscripts. In this case we also add the same superscript to decompose a boundary divisor D_Γ or D_h, e.g.,

$$D_\Gamma \;=\; D_\Gamma^{\mathrm{hyp}} \cup D_\Gamma^{\mathrm{odd}} \cup D_\Gamma^{\mathrm{even}} \quad \text{with} \quad D_\Gamma^\bullet \subset \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)^\bullet \tag{41}$$

as a disjoint union, where $\bullet \in \{\mathrm{hyp}, \mathrm{odd}, \mathrm{even}\}$. Similar decompositions occur if there are only two components, odd and even, or hyperelliptic and non-hyperelliptic.

All the steps of the proof of [CMZ20b, Theorem 1.1] can be performed on each connected component separately. We thus deduce that the component-wise version of Proposition 6.5

$$\frac{\kappa_\mu}{N}\,\mathrm{c}_1\Big(\Omega^d_{\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)^\bullet}(\log D)\Big) \;=\; \psi_{-1} + 12\lambda_1 - [D_h^\bullet] + \sum_{\Gamma\in\mathrm{LG}_1} \ell_\Gamma\Big(\kappa_\mu \frac{N_\Gamma^\perp}{N} - \kappa_{\mu_\Gamma^\perp}\Big)[D_\Gamma^\bullet] \tag{42}$$

holds with $\bullet \in \{\mathrm{hyp}, \mathrm{odd}, \mathrm{even}\}$ for holomorphic signatures μ. As before this formula can be converted to

$$\begin{aligned} \frac{\kappa_\mu}{N}\,\mathrm{c}_1\big(K_{\mathbb{P}\mathrm{MS}(\mu)^\bullet}\big) \;&=\; \psi_{-1} + 12\lambda_1 - \Big(1 + \frac{\kappa_\mu}{N}\Big)[D_h^\bullet] \\ &+ \sum_{\Gamma\in\mathrm{LG}_1}\Big(\frac{\kappa_\mu}{N}(\ell_\Gamma N_\Gamma^\perp - 1) - \ell_\Gamma\kappa_{\mu_\Gamma^\perp}\Big)[D_\Gamma^\bullet] - \frac{\kappa_\mu}{N}\sum_{\substack{\Gamma \text{ is HTB or}\\ \text{HBT or HBB}}}[D_\Gamma^{H,\bullet}] \end{aligned} \tag{43}$$

in $\mathrm{CH}^1(\mathbb{P}\mathrm{MS}(\mu)) = \mathrm{CH}^1(\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu))$ for the (non-logarithmic) canonical bundle. Here $D_\Gamma^{H,\bullet}$ is the (possibly empty) component of D_Γ^{H} in the component indicated by $\bullet \in \{\mathrm{hyp}, \mathrm{odd}, \mathrm{even}\}$. We remark that $D_\Gamma^\bullet$ and $D_\Gamma^{H,\bullet}$ can be further reducible due to connected components of the strata in each level of Γ and also due to the equivalence classes of prong-matchings of the multi-scale differentials they encode. However for our purpose we do not need to classify their irreducible components.

The connected components of strata of meromorphic differentials have been classified by Boissy ([Boi15]). These connected components are similarly distinguished by the spin and hyperelliptic structures, with the only exception in genus one where the distinction is given by divisors of the gcd of the entries in μ (also called the rotation or torsion numbers, see [CC14] and [CG21]). We can analogously add the corresponding superscripts in order to apply Proposition 6.5 and Proposition 6.2 to each connected component of a meromorphic stratum.

6.3. **Divisors of Brill–Noether type.** Recall that two kinds of effective divisors in $\overline{\mathcal{M}}_g$ were used by Harris and Mumford in [HM82] and [Har84] which depend on the parity of g. We proceed similarly. First for g odd, consider the admissible cover compactification of the locus

$$\begin{aligned}\widetilde{BN}_g \;&=\; \{X \in \mathcal{M}_g \,:\, X \text{ has a } \mathfrak{g}^1_k\}, \qquad (k=(g+1)/2)\\ &=\; \{X \in \mathcal{M}_g \,:\, \text{there is a cover } \pi\colon X \to \mathbb{P}^1,\, \deg(\pi)=k\}\,.\end{aligned} \tag{44}$$

This is a classical *Brill–Noether divisor* by the description using linear series. We normalize the class of the Brill–Noether divisor computed in [HM82] to be

$$[\mathrm{BN}_g] \;=\; 6\lambda_1 - \frac{g+1}{g+3}\delta_{\mathrm{irr}} - \sum_{i=1}^{[g/2]} \frac{6i(g-i)}{g+3}\delta_i \quad \in \mathrm{CH}^1(\overline{\mathcal{M}}_g)\,.$$

We define the pullback of the Brill–Noether divisor BN_μ (as a $\mathbb{Q}$-divisor) to be the total transform $[\mathrm{BN}_\mu] = f^*[\mathrm{BN}_g]$ where $f\colon \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) \to \overline{\mathcal{M}}_g$ is the forgetful map. To state the class of the pullback Brill–Noether divisor we use the following notation. For an edge e in a level graph Γ we write $e \mapsto \Delta_i$ if contracting all edges of Γ but e results in a graph of compact type parametrized by the boundary divisor Δ_i in $\overline{\mathcal{M}}_g$. We write $e \mapsto \Delta_{\mathrm{irr}}$ if the edge is non-separating, equivalently if the contraction results in a graph parameterizing irreducible one-nodal curves.

Lemma 6.7. *Let $g \geq 3$ be odd. If the stratum $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is connected then the class of the Brill–Noether divisor in $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$*

$$[\mathrm{BN}_\mu] \;=\; 6\lambda_1 - \frac{g+1}{g+3}[D_h] - \sum_{\Gamma\in\mathrm{LG}_1} b_\Gamma [D_\Gamma]$$

where

$$b_\Gamma \;=\; \ell_\Gamma\Big(\sum_{i=1}^{[g/2]} \sum_{\substack{e\in E(\Gamma)\\ e\mapsto\Delta_i}} \frac{6i(g-i)}{(g+3)p_e} + \sum_{\substack{e\in E(\Gamma)\\ e\mapsto\Delta_{\mathrm{irr}}}} \frac{g+1}{(g+3)p_e}\Big)$$

is an effective divisor class.

If the stratum is disconnected and $\bullet \in \{\mathrm{odd}, \mathrm{even}\}$ denotes a non-hyperelliptic component then similarly

$$[\mathrm{BN}^\bullet_\mu] \;=\; 6\lambda_1 - \frac{g+1}{g+3}[D^\bullet_h] - \sum_{\Gamma\in\mathrm{LG}_1} b_\Gamma [D^\bullet_\Gamma]$$

is an effective divisor class.

Proof. It follows from [Bud21, Theorem 1.1] that for odd g and any non-hyperelliptic connected component of a stratum $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ not every curve parameterized therein admits a $\mathfrak{g}^1_k$ for $k = (g+1)/2$.[5] This implies that the pullback of the Brill–Noether divisor to such strata (components) is an effective divisor. To compute (c times) its class we can perform the same local computation as in the proof of Lemma 6.4. □

We remark that for special Γ, e.g., if the underlying curves parameterized by a vertex of Γ have gonality much smaller than expected, then the total transform BN_μ can contain the corresponding boundary divisor D_Γ, which can be subtracted from BN_μ to make the remaining effective divisor class more extremal.

Second, to cover g even we need two other types of effective divisors. For g even Harris used in [Har84] the closure of

$$\widetilde{\mathrm{Hur}}_g \;=\; \{X \in \mathcal{M}_g \,:\, \text{there is a cover } \pi\colon X \to \mathbb{P}^1,\ \deg(\pi) = (g+2)/2, \\ \pi \text{ has a point of multiplicity three}\,\} \tag{45}$$

where multiplicity being m means locally the cover is given by $x \mapsto x^m$, i.e., being of ramification order $m-1$. We normalize the class of the Hurwitz divisor computed in loc. cit. to be

$$[\mathrm{Hur}_g] \;=\; 6\lambda_1 - \frac{3g^2+12g-6}{(g+8)(3g-1)}\delta_{\mathrm{irr}} - \sum_{i=1}^{g/2} \frac{6i(g-i)(3g+4)}{(g+8)(3g-1)}\delta_i\,.$$

Similarly to the case above we let $[\mathrm{Hur}_\mu] = f^*[\mathrm{Hur}_g]$ be the pullback *Hurwitz divisor class*.

Lemma 6.8. *For even $g \geq 6$, the Hurwitz divisor Hur_μ is an effective divisor in every connected $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ and in every non-hyperelliptic component of disconnected $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. For $g = 4$, Hur_μ is an effective divisor in every connected $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ and in every odd spin component of disconnected $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$.*

Moreover, the class of the Hurwitz divisor in a connected $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is

$$[\mathrm{Hur}_\mu] \;=\; 6\lambda_1 - \frac{3g^2+12g-6}{(g+8)(3g-1)}[D_h] - \sum_{\Gamma\in\mathrm{LG}_1} h_\Gamma[D_\Gamma]$$

where

$$h_\Gamma \;=\; \ell_\Gamma\Big(\sum_{i=1}^{g/2}\sum_{\substack{e\in E(\Gamma)\\ e\mapsto\Delta_i}} \frac{6i(g-i)(3g+4)}{(g+8)(3g-1)p_e} + \sum_{\substack{e\in E(\Gamma)\\ e\mapsto\Delta_{\mathrm{irr}}}} \frac{3g^2+12g-6}{(g+8)(3g-1)p_e}\Big)$$

and the same expression holds for the spin components of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ with D_Γ decorated by $\bullet \in \{\mathrm{odd}, \mathrm{even}\}$ respectively.

[5] Strictly speaking [Bud21] only considered the strata of holomorphic differentials. However the same argument works for the meromorphic case as well by merging all zeros and poles and specializing to the minimal strata, with the exception for $\mu = (2g-2+m, -m)$ with $m > 1$ where the zero and pole cannot be merged due to the GRC. For the exceptional case one can still argue as in loc. cit. by taking a Brill–Noether general curve in $\mathbb{P}\Omega\mathcal{M}_{g-1}(2g-4)^{\mathrm{nonhyp}}$ union an elliptic tail in $\mathbb{P}\Omega\mathcal{M}_1(2-2g, 2g+2-m, -m)$.

Proof. The divisor class calculation is the same as in the proof of Lemma 6.4. In order to prove that no strata (components) are contained in Hur_μ in the claimed range, by merging zeros and poles it suffices to prove it for the minimal strata, i.e., for $(2g-2)^{\mathrm{odd}}$ with even $g \geq 4$ and for $(2g-2)^{\mathrm{even}}$ with even $g \geq 6$ (with the only exception for $\mu = (2g-2+m, -m)$ with $m > 1$ since in this case the zero and pole cannot be merged due to the GRC, which we will treat separately at the end). We will exhibit a (boundary) point in each case that is not contained in Hur_μ.

Consider first the odd spin minimal strata. Take a multi-scale differential $(X, \boldsymbol{\omega})$ consisting of an elliptic curve $(E, p_1, \ldots, p_g) \in \mathbb{P}\Omega\mathcal{M}_1(0^g)$ union a rational curve $(R, q_1, \ldots, q_g, z) \in \mathbb{P}\Omega\mathcal{M}_0(-2^g, 2g-2)$ at g nodes by identifying $p_i \sim q_i$ for all i. The GRC is automatically satisfied by the Residue Theorem on R. Since all prongs are one at the nodes, the spin parity of $(X, \boldsymbol{\omega})$ (via the Arf invariant) is equal to the parity of the flat torus E which is odd. Therefore, $(X, \boldsymbol{\omega})$ is a boundary point of the odd spin minimal stratum. We claim that for general positions of p_i and q_j in E and R, the union X is not contained in the Hurwitz divisor Hur_g for even $g \geq 4$. To see it, consider an admissible cover of degree $g/2+1$ on X. Since the number of nodes between E and R is $g > g/2+1$, E and R must map to the same target $\mathbb{P}^1$-component C, such that each pair p_i and q_i is contained in the same fiber. If all ramification points in E and R are simple, by Riemann–Hurwitz their total number is g. Hence the total number of distinct branch points and image points of p_i, q_i in C is (at most) $2g$. However if the admissible cover has a triple point, then this number drops to (at most) $2g-1$. In other words, the parameter space of such admissible covers with a triple point restricted to E and R has dimension bounded by $\dim \mathcal{M}_{0,2g-1} = 2g-4$. On the other hand, the parameter space of E union R is $\mathcal{M}_{1,g} \times \mathcal{M}_{0,g}$ with dimension $2g-3 > 2g-4$. We thus conclude that a general union of E and R does not admit a cover of degree $g/2+1$ with a triple point.

Next for the even spin minimal strata, take a multi-scale differential $(X, \boldsymbol{\omega})$ consisting of three components, a rational curve $(R, q_1, \ldots, q_g, z) \in \mathbb{P}\Omega\mathcal{M}_0(-2^g, 2g-2)$, an elliptic curve $(E, p_1, \ldots, p_{g-1}) \in \mathbb{P}\Omega\mathcal{M}_1(0^{g-1})$ and another elliptic curve $(E', p_g) \in \mathbb{P}\Omega\mathcal{M}_1(0)$ by identifying $p_i \sim q_i$ for all i. The GRC requires that the residue of ω_R at q_g is zero, which can be satisfied by choosing a special position of z with respect to a general choice of $q_1, \ldots, q_g$ in R. Since all prongs are one at the nodes, the spin parity of $(X, \boldsymbol{\omega})$ is equal to the sum of the parities of the flat tori E and E' which is even. Therefore, $(X, \boldsymbol{\omega})$ is a boundary point of the even spin minimal stratum. Since the number of nodes between E and R is $g-1 > g/2+1$ for $g \geq 6$, one can argue similarly as in the preceding paragraph to show that a general such X is not contained in the Hurwitz divisor Hur_g for even $g \geq 6$.

For the exceptional case $\mu = (2g-2+m, -m)$ with $m > 1$, we can put the unique zero and pole in the rational component R in the above constructions, and the same arguments still work through.

We remark that the discrepancy of the genus bounds for the odd and even spin cases is necessary, because every $(X, z) \in \mathbb{P}\Omega\mathcal{M}_4(6)^{\mathrm{even}}$ admits a triple cover by the linear system $|3z|$ with a triple point at z, and hence the even spin minimal stratum in genus four maps entirely into the Hurwitz divisor. □

In order to show the statement of Theorem 1.4 for even genera minimal strata, the Hurwitz divisor, whose class was computed in Lemma 6.8, is sufficient for all genera apart from $g = 14$. For this special case, in order to show that $\mathbb{P}\mathrm{MS}(26)$ is of general type, we will need to use a pointed Brill–Noether divisor which was studied in the proof of [Far09, Theorem 4.9]. We let $d = g/2 + 1$ and define the *divisor of n-fold points* (here for $n = 2$) as the closure in $\overline{\mathcal{M}}_{g,1}$ of

$$\mathrm{NF}_{g,1} \;=\; \{(X,p) \in \mathcal{M}_{g,1} : \exists \mathcal{L} \in W^1_d(X) : h^0(X, \mathcal{L}(-2p)) \geq 1\}\,.$$

In loc. cit. the class of this divisor was computed to be[6]

$$\begin{aligned}[][\mathrm{NF}_{g,1}] \;&=\; \frac{12}{(g+1)(g-2)}\binom{g-2}{g/2}\frac{g+3}{6}[\mathrm{BN}_g] + \frac{6}{g(g-1)(g+1)}\binom{g}{g/2+1}[\mathrm{W}] \\ &=\; \frac{2(g+3)}{(g+1)(g-2)}\binom{g-2}{g/2}\cdot\Big([\mathrm{BN}_g] + \frac{12}{(g+3)(g+2)}[\mathrm{W}]\Big) \end{aligned} \tag{46}$$

where $[\mathrm{BN}_g]$ is the pullback of the (normalized) Brill–Noether divisor class from $\overline{\mathcal{M}}_g$ and W is the Weierstrass point divisor. The class of W in $\overline{\mathcal{M}}_{g,1}$ is given by

$$[\mathrm{W}] = -\lambda_1 + \binom{g+1}{2}\psi - \sum_{j=1}^{g-1}\binom{g-j+1}{2}\delta_{j;1}$$

where $\delta_{j;1}$ is the class of the boundary divisor parameterizing curves of compact type with a component of genus j that carries the marked point and another unmarked component of genus $g - j$.

We consider the class given by the rescaling of the pullback $f_1^*\,\mathrm{NF}_{g,1}$ having the same λ-coefficient 6 as BN_μ, where $f_1 \colon \mathbb{P}\Xi\overline{\mathcal{M}}_{g,1}(2g-2) \to \overline{\mathcal{M}}_{g,1}$ is the forgetful map.

Lemma 6.9. *Consider even $g \geq 4$ and let $\mathrm{NF}_{(2g-2)} = e^{-1}\cdot f_1^*\,\mathrm{NF}_{g,1}$ where $e = \frac{2(g^2+3g-1)}{(g-1)(g^2-4)}\binom{g-2}{g/2}$. Then the class of $\mathrm{NF}_{(2g-2)}$ is effective for every non-hyperelliptic component of the stratum $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,1}(2g-2)$ for even $g \geq 6$ and for the odd spin component $\mathbb{P}\Xi\overline{\mathcal{M}}_{4,1}(6)^{\mathrm{odd}}$ in $g = 4$.*

Proof. To prove the claim of effectiveness, we need to show that a general $(X, z) \in \mathbb{P}\Xi\overline{\mathcal{M}}_{g,1}(2g-2)^{\mathrm{nonhyp}}$ does not admit a $\mathfrak{g}^1_d$ with multiplicity at least two at z where $d = (g+2)/2$. This was indeed verified in [Bud21, Proposition 3.1]. Again the discrepancy of the genus bounds for the odd and even spin cases is necessary, as every $(X, z) \in \mathbb{P}\Xi\overline{\mathcal{M}}_{4,1}(6)^{\mathrm{even}}$ admits a triple cover induced by the linear system $|3z|$ which is ramified at z. □

The class of $\mathrm{NF}_{(2g-2)}$ can be explicitly computed using (46) and (39).

7. Generalized Weierstrass divisors

There are few effective divisors that are directly defined in the strata of Abelian differentials other than those arising as a pullback from $\overline{\mathcal{M}}_{g,n}$. In this section we directly construct a series of such divisors and compute their classes. These divisors generalize the classical divisor of Weierstrass points in $\overline{\mathcal{M}}_{g,1}$. Despite that they can be defined for both holomorphic and meromorphic signatures, since in this paper we only apply them to certain holomorphic strata, we limit their definition to the *holomorphic* case, and leave the meromorphic case to future work.

In the sequel we will mainly work with the 'middle' case of the generalized Weierstrass divisors, where we use as weights $\mu/2$. Since this tuple is not always

[6]The factor $6/(g + 3)$ compensates the different normalizations of the Brill–Noether divisor class here and in [Far09].

integral, the generalized Weierstrass divisor class associated to it is not obviously effective. In Section 7.4 we discuss the quality of approximation by actual effective classes for generalized Weierstrass divisors given by rounding $\mu/2$.

Finally, with all divisors in place, we give in Section 7.5 a refinement of the general type criterion from Proposition 1.3.

7.1. **The divisor class.** Fix a holomorphic signature $\mu = (m_1, \dots, m_n)$, i.e., with $m_i \geq 0$. If the corresponding stratum $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ is not connected, we suppose moreover that in this section $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ denotes the multi-scale compactification for the connected component of *odd spin*. All the other components (even spin and hyperelliptic) are disregarded for the construction of generalized Weierstrass divisors.

Consider a partition $\alpha = (\alpha_1, \dots, \alpha_n)$ of $g-1$ such that $0 \leq \alpha_i \leq m_i$ for all i. Set-theoretically, the *generalized Weierstrass divisor* associated to α in the interior of a stratum of type μ is given by

$$W_\mu(\alpha) \;=\; \Big\{(X, \mathbf{z}, \omega) \in \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu) \,:\, h^0\Big(X, \sum_{i=1}^n \alpha_i z_i\Big) \geq 2\Big\}\,. \tag{47}$$

By Riemann–Roch and duality we deduce that

$$h^0\Big(X, \sum_{i=1}^n \alpha_i z_i\Big) \;=\; h^0\Big(X, \omega_X - \sum_{i=1}^n \alpha_i z_i\Big) \;=\; h^0\Big(X, \sum_{i=1}^n (m_i - \alpha_i) z_i\Big)$$

from the signature of the stratum, where ω_X denotes the dualizing line bundle of X. Geometrically speaking, the tautological section ω gives a section of the linear system $|\omega_X - \sum_{i=1}^n \alpha_i z_i|$ in the definition of $W_\mu(\alpha)$, and we thus consider the locus where the linear system has extra sections. This viewpoint can realize the generalized Weierstrass divisor as a degeneracy locus and provide it a scheme structure as follows.

Let ω_{rel} be the relative dualizing bundle of the universal curve $\pi\colon \mathcal{X} \to \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ and the $S_i \subset \mathcal{X}$ are the marked sections. Let moreover $\mathcal{H} = \pi_*(\omega_{\mathrm{rel}})$ be the Hodge bundle over $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ and $\mathcal{O}(-1)$ its tautological subbundle. Let $\mathcal{F}_\alpha$ be the bundle whose fiber over $(X, \mathbf{z}, \boldsymbol{\omega})$ is canonically given by $H^0\left(\omega_X/\omega_X(-\sum_{i=1}^n \alpha_i z_i)\right)$ i.e.,

$$\mathcal{F}_\alpha \;=\; \pi_*\Big(\omega_{\mathrm{rel}}/\omega_{\mathrm{rel}}\Big(-\sum_{i=1}^n \alpha_i S_i\Big)\Big).$$

Both $\mathcal{H}/\mathcal{O}(-1)$ and $\mathcal{F}_\alpha$ are vector bundles of rank $g-1$ (for $\mathcal{F}_\alpha$ this follows from the long exact sequence associated with the inclusion $\omega_{\mathrm{rel}}(-\sum_{i=1}^n \alpha_i S_i) \hookrightarrow \omega_{\mathrm{rel}}$ or from the interpretation as the sheaf of principal parts of length α_i at the points z_i). Taking principal parts to order α_i at each of the points z_i defines a bundle map $\mathcal{H} \to \mathcal{F}_\alpha$, which fiberwise is the map $H^0(X, \omega_X) \to H^0(X, \omega_X/\omega_X(-\sum_{i=1}^n \alpha_i z_i))$. Since ω has vanishing order (at least) $m_i \geq \alpha_i$ at each z_i, the above bundle map factors through the following bundle map

$$\phi\colon \mathcal{H}/\mathcal{O}(-1) \to \mathcal{F}_\alpha\,. \tag{48}$$

We define the substack $\widetilde{W}_\mu(\alpha)$ to be the degeneracy locus of ϕ, i.e., the *generalized Weierstrass divisor*

$$\widetilde{W}_\mu(\alpha) \;=\; \{\mathrm{rank}(\phi) < g-1\} \subset \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)\,. \tag{49}$$

To express its divisor class we define

$$\vartheta \;:=\; \vartheta_{\mu,\alpha} \;=\; \sum_{i=1}^{n} \frac{\alpha_i(\alpha_i+1)}{2(m_i+1)} \tag{50}$$

for $\mu = (m_1,\dots,m_n)$ and $\alpha = (\alpha_1,\dots,\alpha_n)$. The quantities $\vartheta^{\perp} := \vartheta_{\mu_\Gamma^{\perp},\alpha_\Gamma^{\perp}}$ and $\vartheta^{\top} := \vartheta_{\mu_\Gamma^{\top},\alpha_\Gamma^{\top}}$ are similarly defined for the bottom and top level strata of Γ, *but* $\alpha_\Gamma^{\perp}$ *and* $\alpha_\Gamma^{\top}$ *assigns value zero to each leg associated with an edge* $e \in E(\Gamma)$.[7] With this convention we also have

$$\vartheta_{\mu_\Gamma^{\perp},\alpha_\Gamma^{\perp}} + \vartheta_{\mu_\Gamma^{\top},\alpha_\Gamma^{\top}} \;=\; \vartheta_{\mu,\alpha}\,.$$

Proposition 7.1. *The substack $\widetilde{W}_\mu(\alpha)$ is an effective divisor in $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. The class of this generalized Weierstrass divisor is given by*

$$\begin{aligned}[\widetilde{W}_\mu(\alpha)] \;=\;& \frac{12+12\vartheta_{\mu,\alpha}-\kappa_\mu}{\kappa_\mu}\lambda_1 - \frac{1+\vartheta_{\mu,\alpha}}{\kappa_\mu}[D_h] \\ &- \sum_{\Gamma\in \mathrm{LG}_1} \ell_\Gamma\Big(\frac{\kappa_{\mu_\Gamma^{\perp}}}{\kappa_\mu}(1+\vartheta_{\mu,\alpha}) - \vartheta_{\mu_\Gamma^{\perp},\alpha_\Gamma^{\perp}}\Big)[D_\Gamma]\,.\end{aligned}$$

The degeneracy locus $\widetilde{W}_\mu(\alpha)$ in general contains extra boundary divisors and we estimate the boundary contributions in the next subsection.

Proof of Proposition 7.1. We first show that $\widetilde{W}_\mu(\alpha)$ is an effective divisor. The setup as degeneracy locus implies that it has (local) codimension at most one everywhere. Hence it suffices to exhibit a (boundary) point not contained in $\widetilde{W}_\mu(\alpha)$.

Consider the boundary divisor D_Γ for $\Gamma \in \mathrm{LG}_1$ consisting of curves X of compact type with a differential $(X_0,q,\omega_0) \in \Omega\mathcal{M}_{g,1}(2g-2)^{\mathrm{odd}}$ on top level, attached via the node q to a rational tail $(R,\mathbf{z},q,\omega_1) \in \Omega\mathcal{M}_{0,n+1}(\mu,-2g)$ carrying all the marked points. This is always possible, including the case of disconnected strata thanks to our standing odd spin hypothesis (since the rational tail has even spin and the parity of the spin is additive for compact type divisors).

In a neighborhood of this boundary divisor we consider the (twisted) line bundle

$$\mathcal{K} \;=\; \mathcal{O}_{\mathcal{X}}\Big(\sum_{i=1}^{n}\alpha_i S_i + (g-1)\mathcal{X}^{\perp}\Big) \tag{51}$$

where $\mathcal{X}^{\perp}$ is the lower level component of the universal family $\mathcal{X}$. Since over the interior the bundle $\pi_*(\mathcal{K})/\mathcal{O}(-1)$ is the kernel of the map ϕ, it suffices to show that on the fiber X over a general point of D_Γ, the bundle $\mathcal{K}$ has a one-dimensional space of sections (i.e., spanned by the tautological section $\boldsymbol{\omega}$ only). The restriction of $\mathcal{K}$ to the fiber X pulled by to its irreducible components gives the bundle $K_0 = \mathcal{O}_{X_0}((g-1)q)$ on X_0 and the degree zero bundle $K_1 = \mathcal{O}_R((1-g)q + \sum_i \alpha_i z_i)$ on X_1 (as can be seen by twisting $\mathcal{K}$ by $(1-g)$ times a fiber class before restricting). Note that for general $(X_0,q) \in \Omega\mathcal{M}_{g,1}(2g-2)^{\mathrm{odd}}$ we have $h^0(X_0,K_0) = h^0(X_0,K_0(-q)) = 1$ (see [Bul13]), which implies that every section of K_0 vanishes at q. Moreover $K_1 \cong \mathcal{O}_{\mathbb{P}^1}$, hence any section of K_1 vanishing at q (in order to glue with sections of K_0) must be identically zero on R. We thus conclude that $h^0(X,\mathcal{K}|_X) = h^0(X_0,K_0) = 1$. In summary, we have exhibited a (boundary)

[7]This is contrary to how edges are treated in the definition of $\kappa_{\mu_\Gamma^{\perp}}$ in Section 6.

point not contained in $\widetilde{W}_\mu(\alpha)$. Consequently $\widetilde{W}_\mu(\alpha)$ is an actual effective divisor in $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$.

In view of the next step we compute the first Chern class of $\mathcal{F}_\alpha$. Suppose $\alpha_1 > 0$ and define α' by decreasing α_1 in α by 1. Then there is an exact sequence (see [EH16, Theorem 11.2 (d)])

$$0 \to \sigma_1^*(\omega_{\mathrm{rel}})^{\alpha_1} \to \mathcal{F}_\alpha \to \mathcal{F}_{\alpha'} \to 0 \tag{52}$$

where σ_1 is the map from the base to the section Z_1 in the universal family. It implies that

$$c_1(\mathcal{F}_\alpha) - c_1(\mathcal{F}_{\alpha'}) \;=\; \alpha_1 \psi_{z_1}$$

and thus, proceeding inductively with all marked points, that

$$c_1(\mathcal{F}_\alpha) \;=\; \sum_{i=1}^n \frac{\alpha_i(\alpha_i+1)}{2}\psi_i\,. \tag{53}$$

To compute the class of the generalized Weierstrass divisor we apply the Porteous formula and obtain that

$$\begin{aligned}
[\widetilde{W}_\mu(\alpha)] \;&=\; c_1(\mathcal{F}_\alpha) - c_1(\mathcal{H}/\mathcal{O}(-1)) \\
&=\; \sum_{i=1}^n \frac{\alpha_i(\alpha_i+1)}{2}\psi_i - \lambda_1 + \xi \\
&=\; \Big(1 + \sum_{i=1}^n \frac{\alpha_i(\alpha_i+1)}{2(m_i+1)}\Big)\xi - \lambda_1 + \sum_{\Gamma\in \mathrm{LG}_1} \ell_\Gamma \sum_{i\in\Gamma^\perp} \frac{\alpha_i(\alpha_i+1)}{2(m_i+1)}[D_\Gamma] \\
&=\; (1+\vartheta_{\mu,\alpha})\xi - \lambda_1 + \sum_{\Gamma\in\mathrm{LG}_1} \ell_\Gamma \vartheta_{\mu_\Gamma^\perp,\alpha_\Gamma^\perp}[D_\Gamma]
\end{aligned}$$

using (38) (here $m_i \geq 0$) and the definition of $\vartheta_{\mu,\alpha}$. By (35) this agrees with the formula we claimed. $\square$

7.2. **The twisted version.** To reduce the boundary contribution in the generalized Weierstrass divisor we replace the bundle map (48) by a twisted version. For simplicity of notation, in the sequel we 'pretend' that there is only one boundary divisor D_Γ, i.e., we work locally in its neighborhood so that in codimension-one there is no other boundary divisor seen. This will improve the boundary coefficient in Proposition 7.1 for this particular Γ. To obtain the global improvement for all D_Γ we can just twist simultaneously by the divisors $V = V_\Gamma$ constructed in the sequel.

We work over a relatively minimal semi-stable model with smooth total space of the universal family $\mathcal{X} \to \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ near D_Γ, i.e., for a node of local type $xy = t^a$ with $a > 1$, we blow it up by inserting $a-1$ semi-stable rational bridges to make the resulting new nodes smooth in the universal family. Let X_i for $i \in I = I_\Gamma$ be the irreducible components of the vertical divisors (including the rational bridges) and let $V = \sum_{i\in I} s_i X_i$ be an effective vertical Cartier divisor supported on some components of the central fiber, with chosen multiplicities $s_i \geq 0$ on each component. Moreover, we require that $s_i = 0$ for all top level components, $s_i \leq \ell_\Gamma$ for all bottom level components, and $s_i \leq kp_e$ if X_i is the k-th rational bridge from the upper end of e to the lower end of e after blowup.

We define the twisted relative dualizing line bundle

$$\mathcal{L} = \omega_\pi(-V)$$

and the associated bundles on $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ given by

$$\mathcal{H}^{\mathcal{L}} = \pi_*\mathcal{L} \quad \text{and} \quad \mathcal{F}^{\mathcal{L}}_{\alpha} = \pi_*\Big(\mathcal{L}/\mathcal{L}\Big(-\sum_{i=1}^{n}\alpha_i Z_i\Big)\Big)\,.$$

Consider the evaluation map as before

$$\phi_{\mathcal{L}}\colon \mathcal{H}^{\mathcal{L}}/\mathcal{O}(-1) \to \mathcal{F}^{\mathcal{L}}_{\alpha}\,. \tag{54}$$

By construction of the multi-scale space the tautological form $\boldsymbol{\omega}$ has vanishing order ℓ_Γ on $\mathcal{X}_\perp$ and vanishing order kp_e on the k-th rational bridge after blowing up a node with prong p_e (also see [Che17a, Section 4] from the twisting viewpoint). The assumption on the range of the twisting coefficients s_i thus ensures that $\mathcal{O}(-1)$ is a sub-bundle of $\mathcal{H}^{\mathcal{L}}$, and hence $\phi_{\mathcal{L}}$ is well-defined by the assumption on the range of α_i.

We denote by $\widetilde{W}_{\mathcal{L}}$ the degeneracy locus of this map, called the *twisted generalized Weierstrass divisor* associated to the twisted relative dualizing line bundle $\mathcal{L}$. Note that $\mathcal{H}^{\mathcal{L}}$ is locally free of rank g away from the codimension-two locus where two or more boundary divisors meet. (In fact, $\mathcal{H}^{\mathcal{L}}$ is torsion-free, and hence locally free over any discrete valuation ring transverse to the boundary. The claim on the rank follows by considering the interior. Away from that codimension-two locus we are complex-analytically locally in a product situation and can apply the DVR-argument.) Working away from this codimension-two locus is sufficient to compute divisor classes by using Porteous' formula.

We need a Grothendieck–Riemann–Roch (GRR) computation before we can fully exploit Porteous' formula.

Lemma 7.2. *Suppose $V = \sum_{i\in I} s_i X_i$ is effective and does not contain an entire fiber, i.e., $s_i \geq 0$ for all i and at least one of s_i is zero. Then $\pi_*\mathcal{O}_{\mathcal{X}}(V) = \mathcal{O}_B$ and*

$$c_1(\mathcal{H}^{\mathcal{L}}) \;=\; \lambda_1 + \frac{1}{2}\pi_*\big([V]^2 - c_1(\omega_\pi)\cdot[V]\big)\,.$$

Proof. For the first statement, note that if all s_i are zero then $\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_B$ since the fibers of $\mathcal{X}$ are connected. If some s_i is positive, consider the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}(V - X_i) \to \mathcal{O}_{\mathcal{X}}(V) \to \mathcal{O}_{X_i}(V) \to 0$$

and its push-forward by π. Twisting the sequence by $-s_i$-times a π-fiber we compute that the degree of $\mathcal{O}_{X_i}(V)$ is $\sum_{e\in X_i\cap X_j}(s_j - s_i)$ where the sum runs over each edge e of X_i. We can choose X_i among those with the largest twisting coefficient such that this degree is negative and hence the push-forward term is zero. Then the fist statement follows from applying induction to $V' = V - X_i$.

For the second statement, denote by $\mathcal{N}$ the nodal locus in $\mathcal{X}$ and let $\gamma = c_1(\omega_\pi)$ for notation simplicity. Then we can apply the first statement, duality and GRR (and the exact sequence $0 \to \Omega_\pi \to \omega_\pi \to \omega_\pi \otimes \mathcal{N} \to 0$ to evaluate $\mathrm{td}^\vee(\Omega_\pi)$) to obtain that

$$\begin{aligned}
&\mathrm{ch}(\mathcal{H}^{\mathcal{L}}) \;=\; \mathrm{ch}(\pi_*\mathcal{L}) - \mathrm{ch}(\pi_*\mathcal{O}_{\mathcal{X}}(V)) \;=\; \mathrm{ch}(\pi_*\mathcal{L}) - \mathrm{ch}(R^1\pi_*\mathcal{L}) \\
&= \pi_*\Big(\mathrm{ch}(\mathcal{L}) \cdot \Big(1 - \frac{\gamma}{2} + \frac{\gamma^2 + \mathcal{N}}{12} + \cdots\Big)\Big) \\
&= \pi_*\Big(\Big(1 + (\gamma - [V]) + \frac{(\gamma - [V])^2}{2} + \cdots\Big) \cdot \Big(1 - \frac{\gamma}{2} + \frac{\gamma^2 + \mathcal{N}}{12} + \cdots\Big)\Big) \\
&= \pi_*\Big(1 + \Big(\frac{\gamma}{2} - [V]\Big) + \Big(\frac{\gamma^2 + \mathcal{N}}{12} + \frac{\gamma^2 - 2\gamma[V] + [V]^2 + \gamma[V] - \gamma^2}{2}\Big) + \cdots\Big) \\
&= \pi_*\Big(1 + \Big(\frac{\gamma}{2} - [V]\Big) + \Big(\frac{\gamma^2 + \mathcal{N}}{12} + \frac{[V]^2 - \gamma[V]}{2}\Big) + \cdots\Big) \\
&= (g-1) + \lambda_1 + \frac{1}{2}\pi_*\big([V]^2 - \gamma[V]\big) + \cdots
\end{aligned}$$

using Noether's formula $\pi_*(\gamma^2 + \mathcal{N})/12 = \lambda_1$, which implies the claimed formula. □

Combining Porteous' formula with Lemma 7.2 and using that (52) turns into

$$0 \to \sigma_1^*(\omega_{\mathrm{rel}}(-V))^{\otimes \alpha_1} \to \mathcal{F}_{\alpha}^{\mathcal{L}} \to \mathcal{F}_{\alpha'}^{\mathcal{L}} \to 0$$

we find that the degeneracy locus $\widetilde{W}_{\mathcal{L}}$ of the map $\phi_{\mathcal{L}}$ in (54) has class

$$\begin{aligned}
[\widetilde{W}_{\mathcal{L}}] &= c_1(\mathcal{F}_{\alpha}^{\mathcal{L}}) - c_1(\mathcal{H}^{\mathcal{L}}) + \xi \\
&= \sum_{i=1}^{n} \frac{\alpha_i(\alpha_i+1)}{2}\psi_i - \Big(\sum_{\substack{i \in I_\Gamma \\ z_j \in X_i}} \alpha_j s_i\Big)[D_\Gamma] - c_1(\mathcal{H}^{\mathcal{L}}) + \xi \\
&= \sum_{i=1}^{n} \frac{\alpha_i(\alpha_i+1)}{2}\psi_i - \Big(\sum_{\substack{i \in I_\Gamma \\ z_j \in X_i}} \alpha_j s_i\Big)[D_\Gamma] + \xi - \lambda_1 - \frac{1}{2}\pi_*[V]^2 + \frac{1}{2}\pi_*\left(c_1(\omega_\pi)[V]\right) \\
&= [\widetilde{W}_{\mu}(\alpha)] - \Big(\sum_{\substack{i \in I_\Gamma \\ z_j \in X_i}} \alpha_j s_i\Big)[D_\Gamma] - \frac{1}{2}\pi_*[V]^2 + \frac{1}{2}\pi_*\left(c_1(\omega_\pi)[V]\right),
\end{aligned}$$

where we recall that we pretend to work with on Γ, instead of writing the sum over all $\Gamma \in \mathrm{LG}_1$. Let ν_i be the number of edges of X_i (e.g., $\nu = 2$ if X_i is a rational bridge) and $\nu_{i,j}$ the number of edges joining X_i and X_j (note that $\nu_{i,i} = 0$ since Γ has no horizontal nodes). Decomposing V into its components and using that $X_iX_i = (X_i - F)X_j$ for a full fiber F of π we find

$$\pi_*[V]^2 \;=\; \Big(-\sum_{i \in I} s_i^2 \nu_i + 2\sum_{i,j \in I} s_i s_j \nu_{i,j}\Big)[D_\Gamma]$$

and

$$\pi_*\left(c_1(\omega_\pi)[V]\right) \;=\; \sum_{i \in I} s_i \pi_*\left(c_1(\omega_\pi)[X_i]\right) \;=\; \Big(\sum_{i \in I} s_i(2g_i - 2 + \nu_i)\Big)[D_\Gamma]\,.$$

The conclusion of this discussion is:

Lemma 7.3. *For integer coefficients s_i of the twisting divisor V we obtain the coefficient difference*

$$\begin{aligned}\Delta_{\mathcal{L}}\widetilde{W}_\Gamma &:= \big([\widetilde{W}_\mu(\alpha)] - [\widetilde{W}_{\mathcal{L}}]\big)_{[D_\Gamma]} \\ &= \sum_{\substack{i\in I\\ z_j\in X_i}} \alpha_j s_i + \sum_{i,j\in I} s_i s_j \nu_{i,j} - \frac{1}{2}\Big(\sum_{i\in I} s_i^2\nu_i + \sum_{i\in I} s_i(2g_i - 2 + \nu_i)\Big)\,. \end{aligned}\tag{55}$$

In what follows we want to maximize this difference by choosing suitable twisting coefficients in the allowed ranges. Denote by $\Delta\widetilde{W}_\Gamma$ the *maximum* of $\Delta_{\mathcal{L}}\widetilde{W}_\Gamma$ among all possible choices of the twisted relative dualizing line bundle $\mathcal{L}$.

We now relabel the vertices of Γ according to levels and single out the irreducible rational components of the central fiber that stem from blowups. Let $X_1,\dots,X_{v^\top}$ be the top level vertices and $Y_1,\dots,Y_{v_\perp}$ the bottom level vertices. Let E_j be the set of edges adjacent to Y_j. For any edge e we denote by $R_e^{(k)}$ the rational bridges that stem from the resolution of the node corresponding to the edge e for $k = 1,\dots,a_e - 1$ where $a_e = \ell_\Gamma/p_e$. Recall that the twisting coefficients s_i are zero for all top level components. We also rename them as σ_j for the bottom level components Y_j and as $s_{e,k}$ for the rational bridges $R_e^{(k)}$, with the convention that $s_{e,0} = 0$ and $s_{e,a_e} = \sigma_j$ for $e\in E_j$. As before we require $\sigma_j \le \ell_\Gamma$ and $s_{e,k}\le kp_e$ for $e\in E_j$.

We introduce the notation $e_{Y_j} = |E_j|$ and

$$m_{Y_j} \;=\; \sum_{z_i\in Y_j} m_i\,, \qquad \alpha_{Y_j} \;=\; \sum_{z_i\in Y_j}\alpha_i\,, \qquad p_{Y_j} \;=\; \sum_{e\in E_j} p_e$$

for the total sum of m_i, the total sum of α_i and the total sum of prongs that are adjacent to Y_j, respectively. The rational bridges do not carry any marked points and the top level gets no twist. It implies that only those α_j on the bottom level contribute to $\Delta_{\mathcal{L}}\widetilde{W}_\Gamma$. We thus obtain that

$$\begin{aligned}\Delta_{\mathcal{L}}\widetilde{W}_\Gamma &= \sum_{j=1}^{v_\perp}\sigma_j\alpha_{Y_j} + \sum_{e\in E}\sum_{k=1}^{a_e-1}(s_{e,k}s_{e,k+1} - s_{e,k}^2) \\ &\quad - \frac{1}{2}\sum_{j=1}^{v_\perp}\sigma_j^2 e_{Y_j} - \frac{1}{2}\sum_{j=1}^{v_\perp}\sigma_j(2g(Y_j) - 2 + e_{Y_j}) \\ &= -\frac{1}{2}\sum_{j=1}^{v_\perp}\sum_{e\in E_j}\sum_{k=1}^{a_e}(s_{e,k}-s_{e,k-1})^2 + \frac{1}{2}\sum_{j=1}^{v_\perp}\sigma_j(2\alpha_{Y_j} - m_{Y_j} + p_{Y_j})\,.\end{aligned}\tag{56}$$

Given σ_j, to maximize the above expression, we minimize the quadratic terms in the first summand by choosing the $s_{e,k}$ nearly equidistant, i.e., roughly $s_{e,k}\sim k\sigma_j/a_e$. Working with this possibly fractional approximation we find that

$$\Delta_{\mathcal{L}}\widetilde{W}_\Gamma \;\gtrsim\; \frac{1}{2\ell_\Gamma}\sum_{j=1}^{v_\perp}\sigma_j\Big(\ell_\Gamma(2\alpha_{Y_j} - m_{Y_j} + p_{Y_j}) - \sigma_j p_{Y_j}\Big)\,.\tag{57}$$

This shows that this is a quadratic optimization problem. In particular for the natural choice $\alpha_i = m_i/2$ the optimal correction term is obtained for an integer approximation of $\sigma_j = \ell_\Gamma/2$.

We now take care of the fractional parts in detail. For $e \in E_j$, dividing σ_j by a_e we write

$$\sigma_j \;=\; q_e a_e + r_e \;=\; (a_e - r_e)q_e + r_e(q_e+1) \tag{58}$$

with $0 \le r_e < a_e$ to compute the numbers of $s_{e,k} - s_{e,k-1}$ equal to q_e and q_e+1, respectively. Note that

$$(a_e - r_e)q_e^2 + r_e(q_e+1)^2 = a_e q_e^2 + 2r_e q_e + r_e = q_e\sigma_j + r_e(q_e+1)\,.$$

We thus obtain from (56) that

$$\Delta_{\mathcal{L}}\widetilde{W}_\Gamma \ge \frac{1}{2}\sum_{j=1}^{v_\perp}\sigma_j(2\alpha_{Y_j} - m_{Y_j} + p_{Y_j}) - \frac{1}{2}\sum_{j=1}^{v_\perp}\sum_{e\in E_j}\big(q_e\sigma_j + r_e(q_e+1)\big)\,. \tag{59}$$

The following lemma optimizes this lower bound if $\alpha_i \sim m_i/2$ for all i.

Lemma 7.4. *The maximum difference between the twisted and untwisted Weierstrass divisors for the coefficient of the boundary divisor D_Γ is at least*

$$\Delta_{\mathcal{L}}\widetilde{W}_\Gamma \;\ge\; \Big\lfloor\frac{\ell_\Gamma}{2}\Big\rfloor\sum_{j=1}^{v_\perp}\Big(\alpha_{Y_j} - \frac{1}{2}m_{Y_j}\Big) + \frac{\ell_\Gamma}{8}\Big(P - P_{-1}\Big) \tag{60}$$

where $P = \sum_{e\in E} p_e$ is the sum of all prongs and $P_{-1} = \sum_{e\in E} 1/p_e$ is the sum of their reciprocals.

Proof. The idea is to show that

$$q_e\sigma_j + r_e(q_e+1) \lesssim \frac{\ell_\Gamma}{4}(p_e + 1/p_e)$$

for an integral choice of $\sigma_j \sim \ell_\Gamma/2$ and apply this to each edge e individually.

First suppose ℓ_Γ is even. In this case we take $\sigma_j = \ell_\Gamma/2$ for all j. If p_e is even, then $a_e = \ell_\Gamma/p_e$ divides σ_j, hence $r_e = 0$ and the above inequality estimate literally holds (even without the term $1/p_e$). If p_e is odd, then $q_e = (p_e-1)/2$, $r_e = a_e/2$, and the inequality estimate becomes an equality.

Next suppose ℓ_Γ is odd. In this case we take $\sigma_j = (\ell_\Gamma - 1)/2$ for all j so that $q_e = (p_e-1)/2$ and $r_e = (a_e-1)/2$. Then we obtain that

$$q_e\sigma_j + r_e(q_e+1) = \frac{\ell_\Gamma}{4}(p_e + 1/p_e) - p_e/2$$

which implies the desired bound since the last term compensates the rounding of σ_j from $\ell_\Gamma/2$ to $(\ell_\Gamma-1)/2$. ☐

7.3. **Improvement for multiple top level components.** Suppose $\Gamma \in \mathrm{LG}_1$ is a level graph with $v^\top > 1$ vertices on top level. Here we show that the degeneracy locus $\widetilde{W}_{\mathcal{L}}$ of the map $\phi_{\mathcal{L}}$ even in the twisted setup contains extra copies of the boundary divisor D_Γ and we estimate the multiplicity. For this purpose it suffices to work over a small disc Δ_t with parameter t transverse to the boundary divisor. Recall that the tautological section $\boldsymbol{\omega}$ vanishes at any top level zero z_i to the zero order m_i (hence at least to order α_i) over Δ and it vanishes on the bottom level to order $\ell = \ell_\Gamma$. Using a plumbing construction we show that besides $\boldsymbol{\omega}$ there exist other such sections:

Proposition 7.5. *There is a subbundle $\mathcal{T} \subset \pi_*(\omega_\pi)$ of rank $v^\top$ whose sections vanish*

- *along the zero sections z_i at any top level vertex to order m_i, and*

- *along the bottom level components of the central fiber to order* t^ℓ.

Proof. By the main theorem of [BCGGM2] the universal family of multi-scale differentials in a neighborhood of the boundary divisor D_Γ can be obtained by plumbing, and thus the family $\pi\colon \mathcal{X} \to \Delta$ over the fixed disc with parameter t and tautological differential ω_t can also be obtained by plumbing. We review the essential steps of the construction in order to show that the plumbing can be performed simultaneously for a $v^\top$-dimensional space of differentials on the central fiber X.

For the plumbing construction each of the nodes of X (corresponding to an edge e) is replaced by the plumbing annulus with differential form

$$\begin{aligned} \mathbb{V}_e \;&=\; \Big\{(u_e,v_e)\in\Delta_\varepsilon^2 \,:\, u_e v_e = t^{\ell/p_e}\Big\} \\ \Omega \;&=\; C\cdot(u_e^{p_e}+t^\ell r)du_e/u_e \;=\; (-C)\cdot t^\ell(v^{-p_e}+r)dv_e/v_e \end{aligned}$$

for ε small and for r, C to be specified. In order to glue in this annulus, we need charts u_e and v_e at the level zero end and level -1 end of the node that put ω_0 into the standard form

$$\omega_0^{(0)} \;=\; u_e^{p_e}du_e/u_e\,, \qquad \omega_0^{(-1)} \;=\; -(v^{-p_e}+t^\ell r)du_e/u_e \tag{61}$$

and add a modification differential $\xi(t)$ locally given by $\xi(t) = t^\ell r du_e/u_e$ supported on level zero to compensate for the missing residue term. The sum $\omega_0 + \xi(t)$ glues with Ω for $C = 1$.

Before we proceed, we remark that for an arbitrary differential η_0 supported on the top level $X_{(0)}$ of the special fiber it is not obvious (and sometimes impossible) to extend it to the plumbed family. In fact in the given chart u_e, in general η_0 is given by an arbitrary power series, hence possibly by a series with arbitrary negative powers in v_e. In this case the existence of a differential $\eta_0^{(-1)}$ on the lower level $X_{(-1)}$ having this prescribed polar part in v_e is unclear. However the situation is better for the following class of differentials.

Let $\mathbf{c} = (c_1,\ldots,c_{v^\top})$ be a tuple of non-zero complex numbers and define the differential $\eta_0(\mathbf{c})$ on $X_0^{(0)}$ to be equal to $c_i\omega_0$ on the i-th component $X_{(0),i}$ of this top level curve for some fixed numbering of these components. For an edge e whose upper end goes to the i-th component, it locally looks like $c_i\cdot(u_e^{p_e}+t^\ell r)du_e/u_e$. Consequently, together with the modification differentials $c_i\xi|_{X_{(0),i}}$ this glues with t^ℓ times a differential $\eta_1(\mathbf{c})$ whose local form at the lower end of the plumbing fixture is given by $-c_i\cdot(v^{-p_e}+r)dv_e/v_e$.

It remains to show that a differential $\eta_1(\mathbf{c})$ on $X_{(-1)}$ with this prescribed principal part near the lower end of each edge exists. By the solution to the Mittag-Leffler problem (e.g., [For91, Theorem 18.11]) it suffices to check that the sum of the residues is zero. Indeed the tautological differential $\boldsymbol{\omega}$ satisfies the global residue condition, i.e., for each i the sum of residues at all the edges connecting to $X_{(0),i}$ is equal to zero. Here all these residues are multiplied by the same constant c_i. Consequently the sum of the residues required in the polar parts of $\eta_1(\mathbf{c})$ is zero, and hence $\eta_1(\mathbf{c})$ exists.

We now take $\mathcal{T}$ to be the subbundle generated by the plumbings of the differentials $(\eta_0(\mathbf{c}),\eta_1(\mathbf{c}))$. Since the top level is just a rescaling of η_0 on each top level vertex the first condition holds, and the second condition also holds as mentioned in the above plumbing process. $\square$

Corollary 7.6. *The degeneracy locus $\widetilde{W}_{\mathcal{L}}$, where $\mathcal{L} = \omega_\pi(-V)$ and V is an effective vertical divisor containing the bottom level components with multiplicity σ, contains the boundary divisor D_Γ with multiplicity at least $(v^\top - 1)(\ell - \sigma)$.*

Proof. Recall that $\widetilde{W}_{\mathcal{L}}$ is defined as the vanishing locus of the determinant of the Porteous matrix with rows indexed by a basis of sections of $\mathcal{L}/\mathcal{O}(-1)$ and columns indexed by the local expansions up to order α_i at the zeros z_i. Consider the $v^\top - 1$ rows corresponding to a basis of sections of $\mathcal{T}/\mathcal{O}(-1)$ given by Proposition 7.5, which is a subbundle of $\mathcal{L}/\mathcal{O}(-1)$. The properties listed in the proposition imply that the entries in the α_i columns for the zero z_i vanish for all z_i on top level, and moreover, that the entries in the remaining columns for z_i on the bottom level are divisible by $t^{\ell-\sigma}$ (since we have already twisted off t^σ for the bottom level in $\mathcal{L}$). Taking the determinant of the matrix thus implies the claim. □

We apply the previously obtained improvements from twisting and from top level to the 'middle case' of the generalized Weierstrass divisor by taking $\alpha = \mu/2$.

Corollary 7.7. *If all zero orders m_i are even in μ, then the class*

$$[W^{\mathrm{mid}}] := w_\lambda^{\mathrm{mid}}(\mu)\lambda - w_{\mathrm{hor}}^{\mathrm{mid}}(\mu)[D_h] - \sum_{\Gamma \in \mathrm{LG}_1} w_\Gamma^{\mathrm{mid}}(\mu)\ell_\Gamma[D_\Gamma] \tag{62}$$

is an effective divisor class where

$$w_\lambda^{\mathrm{mid}}(\mu) \;=\; \frac{12 + \kappa_\mu/2}{\kappa_\mu}\,, \quad w_{\mathrm{hor}}^{\mathrm{mid}}(\mu) \;=\; \frac{1 + \kappa_\mu/8}{\kappa_\mu}\,, \quad w_\Gamma^{\mathrm{mid}}(\mu) \;=\; \frac{1}{2}(v^\top - 1) + \frac{\kappa^\perp}{\kappa_\mu}\,.$$

We do not claim that $[W^{\mathrm{mid}}]$ is the class of the closure of the interior locus $W_\mu(\mu/2)$ defined in (47) at the beginning of this section, as there might exist special boundary divisors D_Γ that can be subtracted further from $[W^{\mathrm{mid}}]$ such that the remaining class is still effective. However, for certain μ and Γ there are evidences for the sharpness of our bound (which we do not discuss here to avoid making the paper too long). With these in mind, we call $[W^{\mathrm{mid}}]$ the *class of the mid-range generalized Weierstrass divisor.*

Proof. For μ even and $\alpha = \mu/2$ we have $\vartheta_{\mu,\alpha} = \kappa_\mu/8$. This converts the expressions from Proposition 7.1 into the desired forms of $w_{\mathrm{hor}}^{\mathrm{mid}}(\mu)$ and $w_\lambda^{\mathrm{mid}}(\mu)$ for the corresponding coefficients. For the D_Γ-coefficient we rewrite the ϑ's and κ's in terms of the top level versions. Then we have

$$8\Big(\frac{\kappa^\perp}{\kappa_\mu}\vartheta_{\mu,\alpha} - \vartheta_{\mu,\alpha}^\perp\Big) = 8\vartheta_{\mu_\Gamma^\top,\alpha_\Gamma^\top} - \kappa_{\mu_\Gamma^\top} \;=\; P_{-1} - P\,.$$

Using the twisted version of the mid-range generalized Weierstrass divisor from Lemma 7.4 as well as the improvement from Corollary 7.6, we conclude that the class with $-\ell_\Gamma[D_\Gamma]$-coefficient equal to

$$\begin{aligned} &\frac{\kappa^\perp}{\kappa_\mu} + \Big(\frac{\kappa^\perp}{\kappa_\mu}\vartheta_{\mu,\alpha} - \vartheta_{\mu,\alpha}^\perp\Big) + \frac{P - P_{-1}}{8} + \frac{\ell - \sigma}{\ell}(v^\top - 1) \\ =\; &\frac{\kappa^\perp}{\kappa_\mu} + \frac{\ell - \sigma}{\ell}(v^\top - 1) \end{aligned}$$

is effective. Since in our setting $\sigma = \ell/2$ or $(\ell - 1)/2$, then $-(\ell - \sigma)/\ell \le -1/2$, hence the class with $-\ell_\Gamma[D_\Gamma]$-coefficient given by $w_\Gamma^{\mathrm{mid}}(\mu)$ is (possibly more) effective. □

7.4. **Odd order zeros and rounding approximations.** For general strata we may still *define* the class $[W^{\rm mid}]$ by the formula (62) in Corollary 7.7. However if the zero orders m_i are not all even, then this class is not obviously effective as $\alpha = \mu/2$ is not an integer tuple. In this case we approximate it by taking the average of rounding up and down.

Let $R(\mu/2)$ be the set of *admissible roundings* for $\mu/2$, defined as follows. For $\alpha \in R(\mu/2)$ we require that $\alpha_i = m_i/2$ if m_i is even, that $\alpha_i \in \{(m_i \pm 1)/2\}$ if m_i is odd, and that the total sum of α_i is $g-1$. That is, we round up and down in precisely half of the cases. Define the effective divisor class

$$[W^{\rm app}] \;=\; |R(\mu/2)|^{-1} \cdot \sum_{\alpha \in R(\mu/2)} [W_\mu(\alpha)] \tag{63}$$

and thus $[W^{\rm app}] = [W^{\rm mid}]$ if all entries are even. To compute the difference of these two classes in general we write

$$[W_\mu(\alpha)] = w_\lambda^\alpha(\mu)\lambda - w_{\rm hor}^\alpha(\mu)[D_h] - \sum_{\Gamma \in {\rm LG}_1} w_\Gamma^\alpha(\mu)\ell_\Gamma[D_\Gamma] \tag{64}$$

and similarly with the upper index by mid or app. We summarize that so far we have computed

$$w_\lambda^\alpha(\mu) \;=\; \frac{12 + 12\vartheta_{\mu,\alpha} - \kappa_\mu}{\kappa_\mu}, \quad w_{\rm hor}^\alpha(\mu) \;=\; \frac{1+\vartheta_{\mu,\alpha}}{\kappa_\mu},$$

$$w_\Gamma^\alpha(\mu) \;\geq\; \frac{\kappa_{\mu_\Gamma^\perp}}{\kappa_\mu}(1+\vartheta_{\mu,\alpha}) - \vartheta^\perp + \Big\lfloor \frac{\ell_\Gamma}{2} \Big\rfloor \sum_{i=1}^{v_\perp} \Big(\frac{2\alpha_{Y_i} - m_{Y_i}}{2\ell_\Gamma} \Big) + \frac{P - P_{-1}}{8} + \frac{1}{2}(v^\top - 1)\,.$$

Lemma 7.8. *Let μ denote holomorphic signatures. For the λ-coefficients we have $w_\lambda^{\rm app}(\mu) \geq w_\lambda^{\rm mid}(\mu)$ and $\lim w_\lambda^{\rm app}(\mu) \geq \lim w_\lambda^{\rm mid}(\mu) = 1/2$ as $g \to \infty$.*

Moreover for the coefficients of $D_{\rm hor}$ we have $w_{\rm hor}^{\rm app}(\mu) \geq w_{\rm hor}^{\rm mid}(\mu)$ and $\lim w_{\rm hor}^{\rm app}(\mu) \geq \lim w_{\rm hor}^{\rm mid}(\mu) = 1/8$ as $g \to \infty$.

Finally for any $\Gamma \in {\rm LG}_1$ and any holomorphic stratum

$$w_\Gamma^{\rm app}(\mu) - w_\Gamma^{\rm mid}(\mu) \;\geq\; \frac{\kappa_{\mu_\Gamma^\perp}}{\kappa_\mu} \sum_{m_i \text{ odd}} \frac{1}{8(m_i+1)} \;-\; \sum_{\substack{z_i \in \Gamma_\perp \\ m_i \text{ odd}}} \frac{1}{8(m_i+1)}.$$

Proof. We define $\theta(a,m) = a(a+1)/2(m+1)$. The key observation is that

$$\frac{1}{2}\Big(\theta\Big(\frac{m-1}{2}, m\Big) + \theta\Big(\frac{m+1}{2}, m\Big)\Big) - \theta\Big(\frac{m}{2}, m\Big) \;=\; \frac{1}{8(m+1)}\,.$$

We apply this to the summands of $\vartheta_{\mu,\alpha}$ and observe that any odd order m_i is rounded up resp. down in $R(\mu/2)$ half of the times. This gives the inequalities for the λ-coefficients and the $D_{\rm hor}$-coefficients. The claim on their limits follows from these inequalities together with the relation $\vartheta_{\mu,\mu/2} = \kappa_\mu/8$ and the fact that $\kappa_\mu \to \infty$ as $g \to \infty$.

We now consider the w_Γ-coefficient. In the comparison between app and mid all the terms involving neither ϑ nor $\vartheta^\perp$ cancel. It implies that

$$w_\Gamma^{\rm app}(\mu) - w_\Gamma^{\rm mid}(\mu) \;\geq\; \frac{\kappa_{\mu_\Gamma^\perp}}{\kappa_\mu} \sum_{m_i \text{ odd}} \frac{1}{8(m_i+1)} \;-\; \sum_{\substack{z_i \in \Gamma_\perp \\ m_i \text{ odd}}} \frac{1}{8(m_i+1)} \tag{65}$$

which yields the desired inequality. ◻

Remark 7.9. Denote by $M_{-1}^{\mathrm{odd}} := \sum_{m_i \mathrm{odd}} \frac{1}{m_i+1}$. Then by the proof of Lemma 7.8 we obtain that

$$w_\lambda^{\mathrm{app}} = w_\lambda^{\mathrm{mid}} + \frac{12M_{-1}^{\mathrm{odd}}}{8\kappa_\mu}\,.$$

In particular, if M_{-1}^{odd} is negligible comparing to the magnitude of κ_μ, then we see that the large genus limits of $w_\lambda^{\mathrm{app}}(\mu)$ and $w_{\mathrm{hor}}^{\mathrm{app}}(\mu)$ coincide with the corresponding limits of the mid-range version, i.e., being $1/2$ and $1/8$ respectively. For instance, this is the case for signatures μ whose number of entries is a constant independent of g.

In the case of equidistributed strata with $\mu = (s^n)$, when s is odd, the above specializes to the equality

$$w_\lambda^{\mathrm{app}} = w_\lambda^{\mathrm{mid}} + \frac{3}{2s^2+4s}\,.$$

7.5. **Refining Proposition 1.3 for strata with two zeros.** For strata of type $\mu = (2m, 2g-2-2m)$ the canonical class of the coarse moduli space (rescaled by the factor $\frac{\kappa_\mu}{N}$) is not given by the right-hand side of (36), due to the ramification of the map from the stack to the coarse moduli space in the interior, as explained in Proposition 2.2. With the help of the following proposition we can apply the 'ample+effective'-criterion Proposition 1.3 formally without worrying about the presence of the ramification divisor, as long as we use effective divisors containing the ramification divisor with suitably high coefficients.

Proposition 7.10. *Let K'_μ denote the right-hand side of (36). Consider the coarse moduli space $\mathbb{P}\mathrm{MS}(\mu)$ of multi-scale differentials of type $\mu = (2m, 2g-2-2m)$ with two labeled zeros. If we can write*

$$K'_\mu - \frac{\kappa_\mu}{N} D_{\mathrm{NC}} \;=\; A + x[B] + y\frac{12}{w_\lambda^{\mathrm{app}}}[W^{\mathrm{app}}] \tag{66}$$

with A an ample divisor class and $B = \mathrm{BN}_\mu$ or $B = \mathrm{Hur}_\mu$ depending on the parity of g, and if moreover $x \geq 0$ and $y > 1/24$, then $\mathbb{P}\mathrm{MS}(\mu)$ is a variety of general type for sufficiently large g.

Proof. Let R be the interior ramification divisor exhibited in Proposition 2.2. This proposition implies that

$$\frac{\kappa_\mu}{N} K_{\mathbb{P}\mathrm{MS}(\mu)} \;=\; K'_\mu - \frac{\kappa_\mu}{N} R\,.$$

Note that the Brill–Noether divisor $\widetilde{\mathrm{BN}}_g$ from (44), the Hurwitz divisor $\widetilde{\mathrm{Hur}}_g$ from (45), and $W_\mu(\alpha)$ for any α contain the locus of hyperelliptic curves, and hence contain R. We write $\widetilde{B} = f^*\widetilde{\mathrm{BN}}_g$ or $\widetilde{B} = f^*\widetilde{\mathrm{Hur}}_g$ depending on the parity of g, where $f\colon \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) \to \overline{\mathcal{M}}_g$. This implies that

$$\frac{\kappa_\mu}{N}(K_{\mathbb{P}\mathrm{MS}(\mu)} - D_{\mathrm{NC}}) \;=\; A + \frac{x}{c}[\widetilde{B} - R] + y\frac{12}{w_\lambda^{\mathrm{mid}}}[W^{\mathrm{mid}} - R] + \Big(\frac{x}{c} + y\frac{12}{w_\lambda^{\mathrm{mid}}} - \frac{\kappa_\mu}{N}\Big)R\,,$$

where c is the (very large) coefficient of rescaling from $\widetilde{B}$ to B (given in [HM82] resp. in [Har84]) and the mid and app versions of W coincide for $\mu = (2m, 2g-2-2m)$ with even entries only. Note that $\frac{\kappa_\mu}{N} \leq 1$ and $\frac{12}{w_\lambda^{\mathrm{mid}}(\mu)} \to 24$ as $g \to \infty$. We thus conclude that the above is a sum of an ample class and an effective class. $\square$

8. Certifying general type

In this section we prove Theorem 1.4, Theorem 1.5 and Theorem 1.6 about the Kodaira dimension of strata with few zeros and equidistributed strata. The general strategy is to apply Proposition 1.3 and its variant Proposition 7.10, using the ample divisor constructed in the proof of Theorem 1.1 and a combination of effective divisors introduced in Section 6 and Section 7.

On one hand we use the generalized Weierstrass divisor minus its extraneous boundary components. For simplicity we work exclusively with the average case $\alpha = \mu/2$ or its nearest integer approximation, i.e., with the class $[W^{\mathrm{app}}]$ discussed in Section 7.4. On the other hand for g odd we use the Brill–Noether divisor class given in Lemma 6.7. For g even the Brill–Noether divisor is replaced by the Hurwitz divisor in Lemma 6.8 (and for the minimal stratum with $g = 14$, it is replaced by the divisor $\mathrm{NF}_{(2g-2)}$ of Lemma 6.9).

Technically, we work with a convex combination such that the λ-coefficient is zero, and all boundary terms will be shown to be *strictly* positive. Then we can subtract a small multiple of the ample class while maintaining the boundary terms positive. More precisely, we consider the sum

$$
\begin{aligned}
&\frac{\kappa_\mu}{N}\Big(c_1\big(K_{\mathbb{P}\mathrm{MS}(\mu)}\big) - D_{\mathrm{NC}}\Big) - y\frac{12\kappa_\mu}{12+12\vartheta^{\mathrm{app}}-\kappa_\mu}[W^{\mathrm{app}}] - (1-y)\cdot 2[\mathrm{BN}_\mu] \\
&\quad = \; s_{\mathrm{hor}}(y)[D_h] + \sum_{\Gamma\in\mathrm{LG}_1} \ell_\Gamma\big(s_\Gamma^{\mathrm{H}}(y)[D_\Gamma^{\mathrm{H}}] + s_\Gamma^{\mathrm{NH}}(y)[D_\Gamma^{\mathrm{NH}}]\big)
\end{aligned}
$$

with symbols defined as follows. The ramified boundary components D_Γ^{H} of the map from the stack to the coarse moduli space have been singled out for the boundary divisors of type HTB, HBT and HBB in Section 2.3 and they are empty otherwise. The components D_Γ^{NH} denote the corresponding complement in D_Γ in each case. The coefficients are

$$
\begin{aligned}
s_{\mathrm{hor}}(y) \;&=\; -1-\frac{\kappa_\mu}{N}+y\frac{12(1+\vartheta^{\mathrm{app}})}{12+12\vartheta^{\mathrm{app}}-\kappa_\mu}+(2-2y)\frac{g+1}{g+3} \\
s_\Gamma^{\bigstar}(y) \;&=\; c_\Gamma^{\bigstar}+y\frac{12w_\Gamma^{\mathrm{app}}(\mu)}{w_\lambda^{\mathrm{app}}(\mu)}+(1-y)b_\Gamma \qquad \text{for } \bigstar\in\{\mathrm{H},\mathrm{NH}\}
\end{aligned}
\tag{67}
$$

where for g odd the contributions in $s_\Gamma^{\bigstar}(y)$ of the canonical bundle and the Brill–Noether divisor are respectively given by

$$
\begin{aligned}
c_\Gamma^{\bigstar} \;&=\; \frac{\kappa_\mu}{N}\Big(N_\Gamma^{\perp}-R_\Gamma^{\bigstar}\Big)-\kappa_{\mu_\Gamma^{\perp}}\,, \\
b_\Gamma \;&=\; \sum_{i=1}^{[g/2]}\sum_{\substack{e\in E(\Gamma)\\ e\mapsto\Delta_i}}\frac{12i(g-i)}{(g+3)p_e}+\sum_{\substack{e\in E(\Gamma)\\ e\mapsto\Delta_{\mathrm{irr}}}}\frac{2(g+1)}{(g+3)p_e}\,,
\end{aligned}
\tag{68}
$$

and where the coefficients of the Weierstrass divisor are summarized in (65) and Lemma 7.8. For g even, we need to replace b_Γ by the corresponding coefficient of $[\mathrm{Hur}_g]$. Here $R_\Gamma^{\bigstar}$ is the (renormalized) contribution of the NC-compensation divisor plus one (coming from the difference between the canonical and the log-canonical class) and plus the contribution of the ramification divisor if $\bigstar = H$:

$$
R_\Gamma^{\bigstar} := \frac{b_{\mathrm{NC}}^{\Gamma}+1+\delta_\Gamma^{\mathrm{H}}}{\ell_\Gamma}\,. \tag{69}
$$

Recall that $b_{\rm NC}^{\Gamma}$ was defined in (28) and $\delta_{\Gamma}^{\rm H} = 1$ if Γ belongs to HBB, HBT or HTB (see Figure 1) and $\bigstar = H$, i.e., if D_Γ contains a ramification divisor of the map to the coarse moduli space, and zero otherwise. Note that $c_{\Gamma}^{\rm H} \leq c_{\Gamma}^{\rm NH}$ and likewise for $s_{\Gamma}^{\rm H}(y)$ for all y. For the purpose of estimates we thus define $s_\Gamma(y) := \min\{s_{\Gamma}^{\rm H}(y), s_{\Gamma}^{\rm NH}(y)\}$ and $c_\Gamma := \min\{c_{\Gamma}^{\rm H}, c_{\Gamma}^{\rm NH}\}$ if Γ belongs to HBB, HBT or HTB, and we need to control this quantity only.

For the definition of strata with few zeros, we refer to the condition (81). As easily seen in its weaker version (82), this in particular implies the condition stated in Theorem 1.5.

The *proof of Theorem 1.5 and Theorem 1.6* can be reduced to showing:

Proposition 8.1. *For all but a finite number of strata with few zeros and even signature, if $y = 1/4 - \varepsilon$ then $s_{\rm hor}(y)$ and $s_\Gamma(y)$ are strictly positive for all Γ, where ε is a constant depending on g defined in (78). Moreover, for strata with two zeros and odd signature, the analogous statement is true for $y = 1/6$.*

For all but finitely many equidistributed strata $\mu = (s^n)$ with few zeros, there is a choice of a positive $y < 1$ such that $s_{\rm hor}(y)$ and $s_\Gamma(y)$ are strictly positive for all Γ.

Similarly the *proof of Theorem 1.4* can be reduced to showing:

Proposition 8.2. *For $g \geq 44$ the coefficients $s_{\rm hor}(0.19)$ and $s_\Gamma(0.19)$ are strictly positive for all Γ. For the range $13 \leq g \leq 43$, the coefficients $s_{\rm hor}(y)$ and $s_\Gamma(y)$ are both strictly positive for all Γ for y given in Figure 7.*

g	13	14	15	16	17	18
$y \in$	$[0.78, 0.79]$	$[0.67, 0.68]$	$[0.59, 0.73]$	$[0.63, 0.66]$	$[0.47, 0.68]$	$[0.54, 0.62]$
g	19	20	30	40	43	44
$y \in$	$[0.39, 0.53]$	$[0.47, 0.59]$	$[0.29, 0.42]$	$[0.22, 0.35]$	$[0.21, 0.37]$	$[0.21, 0.34]$

FIGURE 7. Range of y for showing that minimal strata with odd spin are of general type. For $g = 14$ we use the $\mathrm{NF}_{(2g-2)}$ divisor instead of the Hurwitz divisor $\mathrm{Hur}_{(2g-2)}$ to substitute the Brill–Noether divisor $\mathrm{BN}_{(2g-2)}$.

Proof of Theorem 1.4, Theorem 1.5 and Theorem 1.6. If we assume the claims of Proposition 8.1 and Proposition 8.2, then we can write

$$c_1\big(K_{\mathbb{P}\mathrm{MS}(\mu)}\big) - D_{\rm NC} = C_1 \cdot [W^{\rm app}] + C_2 \cdot 2[\mathrm{BN}_\mu] + E'$$

where C_i are positive constants and E' is a linear combination of all boundary divisors with strictly positive coefficients. (In the previous expression, the Brill–Noether divisor has to replaced by the Hurwitz divisor or the NF divisor for even genera.) Let $A = \lambda_1 + \varepsilon\, \mathrm{c}_1(\mathcal{L}_{\overline{B}} \otimes \mathcal{O}_{\overline{B}}(-D))$ be the ample class constructed in Section 3 (see in particular the introductory paragraphs and Proposition 3.2). Hence by slightly perturbing the coefficient of $[W^{\rm app}]$ we obtain that for δ_1 small enough

$$c_1\big(K_{\mathbb{P}\mathrm{MS}(\mu)}\big) - D_{\rm NC} = C_1' \cdot [W^{\rm app}] + C_2 \cdot 2[\mathrm{BN}_\mu] + \delta_1 A + \delta_2 \lambda_1 + E''$$

with $\delta_2 > 0$ and where E'' is still effective. We can now apply Proposition 1.3 for all strata in the given list, except for strata with two zeros, for which the canonical class is not given by the formula (68) because of the ramification of the map from the stack to the coarse moduli space in the interior, as explained in Proposition 2.2. For strata with two zeros, we can nevertheless apply Proposition 7.10 since the value of y used in Proposition 8.1 is $1/6$, which is greater than $1/24$. $\square$

We have now reduced to prove Proposition 8.1 and Proposition 8.2.

8.1. **A summary of notations.** We work throughout in the stratum with signature $\mu = (m_1, \dots, m_n)$. If μ is a holomorphic signature the unprojectivized dimension of the corresponding stratum is $N = 2g + n - 1$. We let

$$M \;=\; \sum_{i=1}^{n} m_i = 2g-2\,, \qquad M_{-1} \;=\; \sum_{i=1}^{n} \frac{1}{m_i+1}\,. \tag{70}$$

Recall from (34) and (50) that

$$\kappa \;:=\; \kappa_\mu \;=\; \sum_{i=1}^{n} \frac{m_i(m_i+2)}{m_i+1} \;=\; 2g-2+n-M_{-1}\,, \quad \vartheta \;:=\; \vartheta_{\mu,\alpha} \;=\; \sum_{i=1}^{n} \frac{\alpha_i(\alpha_i+1)}{2(m_i+1)}$$

if $\alpha = (\alpha_1, \dots, \alpha_n)$ is a partition of $g-1$ such that $0 \le \alpha_i \le m_i$ for all i.

If Γ is a two-level graph, all these notations have the corresponding meaning for top and bottom levels, giving rise to $N^\top, N^\perp$, to $M^\top, M^\top_{-1}, M^\perp, M^\perp_{-1}$ and to $\kappa^\top := \kappa_{\mu_\Gamma^\top}$ etc, so that

$$\kappa^\perp + \kappa^\top \;=\; \kappa_\mu\,, \qquad \vartheta^\perp + \vartheta^\top = \vartheta\,.$$

Finally, level graphs come with the prongs associated with the edges and we define

$$P = \sum_{e \in E} p_e\,, \qquad P_{-1} = \sum_{e \in E} 1/p_e\,. \tag{71}$$

Lemma 8.3. *For any meromorphic type μ without simple poles we have $\kappa_\mu \ge 0$. Moreover, $\kappa_\mu = 0$ if and only if $\mu = (0, \dots, 0)$ or $\mu = (-m, m-2, 0, \dots, 0)$ for $m \ge 2$. In the remaining cases we have $\kappa_\mu \ge 1/3$.*

Proof. Note that $m_i/(m_i+1) \ge 1/2$ for $m_i > 0$ and $m_i/(m_i+1) > 1$ for $m_i < -1$. If $g \ge 1$, the claim follows since $n - M_{-1} \ge 1/2$ if $\mu \ne (0, \dots, 0)$. Consider the case of $g = 0$, for which $n \ge 3$. We can assume that μ has at least three non-zero entries m_1, m_2, m_3, with at least one positive and one negative, say $m_1 > 0$ and $m_2 < -1$. If $m_3 > 0$, then $\kappa_\mu \ge -2 + 1/2 + 1/2 + 4/3 = 1/3$ (with equality attained for $\mu = (1, 1, -4, 0, \dots, 0)$). If $m_3 < -1$, then $\kappa_\mu > -2 + 1/2 + 1 + 1 > 1/3$. $\square$

We summarize some more parameters that coarsely classify graphs $\Gamma \in \mathrm{LG}_1$. These are the number of edges $E = E_\Gamma$, the number of vertices $v^\top$ on top and $v^\perp$ on the bottom, the number of marked points $n^\top$ on top and $n^\perp$ on the bottom and the genera $g_i^\top$ of the vertices on top and their sum $g^\top = \sum g_i^\top$ and similarly for bottom level.

Finally we recall from Section 5.4 that the definition of D_{NC} in (28), together with the definition of $R := R_\Gamma^\bigstar$ above, gives a bound

$$R \le \frac{1}{2} P_{-1}^{\mathrm{NCT}} + 4 P_{-1}^{\mathrm{EDB}} + P_{-1}^{\mathrm{RBT}} + 2 P_{-1}^{\mathrm{OCT}} + \frac{\delta_\Gamma^{\mathrm{H}}}{\ell_\Gamma} \tag{72}$$

where EDB are *elliptic dumbbells*, compact type edges with one elliptic end which we consider only if Γ has exactly one edge, where RBT are *rational bottom tails*, tails with a rational vertex on bottom level, and OCT abbreviates *other compact* *type* edges. Moreover NCT denotes *non-compact type* edges. We define P_{-1}^{NCT} etc, as in (71), with the sum restricted to the corresponding subset of edges.

Under these notations the Γ-coefficients of the Brill–Noether divisor class are estimated by

$$b_\Gamma \;\geq\; 2\frac{g+1}{g+3}P_{-1}^{\mathrm{NCT}} + 12\frac{g-1}{g+3}P_{-1}^{\mathrm{OCT}} + 12\frac{g-1}{g+3}P_{-1}^{\mathrm{EDB}}\,. \tag{73}$$

8.2. **The strategy for a general stratum.** The horizontal divisor, the analogue of δ_{irr} for $\overline{\mathcal{M}}_g$, is not the main concern here, as suggested by a coarse estimate as follows, which is not restricted to the special strata we consider but holds true in the general case of any holomorphic signature.

Lemma 8.4. *For each $y > 0$ there are at most finitely many holomorphic strata such that $s_{\mathrm{hor}}(y) > 0$ does not hold.*

For the minimal stratum $\mu = (2g-2)$ we have $s_{\mathrm{hor}}(0.19) > 0$ for $g \geq 44$ and $s_{\mathrm{hor}}(y) > 0$ for $12 \leq g \leq 43$ and for y satisfying the lower bound ranges given in Figure 7 (and the lower bound $y > 0.91$ for $g = 12$).

Proof. We start with the case g odd. The coefficient $s_{\mathrm{hor}}(y) > 0$ if and only if

$$y \geq x := \Big(1 + \frac{\kappa}{N} - \frac{2g+2}{g+3}\Big)\Big(\frac{12(1+\vartheta^{\mathrm{app}})}{12+12\vartheta^{\mathrm{app}}-\kappa} - \frac{2g+2}{g+3}\Big)^{-1}.$$

If the entries of μ are even and we use $\alpha = \mu/2$, then $\vartheta^{\mathrm{app}} = \vartheta^{\mathrm{mid}} = \kappa/8$, and hence

$$\frac{12(1+\vartheta^{\mathrm{app}})}{12+12\vartheta^{\mathrm{app}}-\kappa} \;=\; \frac{24+3\kappa}{24+\kappa} \to 3$$

as $g \to \infty$. Since

$$\frac{\kappa}{N} = 1 - \frac{1+\sum_{i=1}^{n}\frac{1}{m_i+1}}{2g+n-1} \;<\; 1\,, \tag{74}$$

the numerator of x is smaller than any positive bound as $g \to \infty$. If the entries of μ are odd, we can use the relation $\vartheta^{\mathrm{app}} = \vartheta^{\mathrm{mid}} + M_{-1}^{\mathrm{odd}}$ shown in the proof of Lemma 7.8 to prove that the denominator of x still converges to a positive constant.

We next deal with the case g even and μ not the minimal stratum, where the expression $s_{\mathrm{hor}}^{\mathrm{app}}(y)$ involves the Hurwitz divisor. Its negative D_h-coefficient is within $O(1/g)$ of the coeefficient of BN_μ. This implies that whenever we claimed above that $s_{\mathrm{hor}}^{\mathrm{app}}(y) > 0$ for a fixed y and all but finitely many strata, the same claim holds for the corresponding sum $s_{\mathrm{hor}}^{\mathrm{app}}(y)$ involving Hur_g.

For the minimal strata, one can check that the Hurwitz divisor gives a smaller $s_{\mathrm{hor}}^{\mathrm{app}}(y)$. By using the expression of $s_{\mathrm{hor}}^{\mathrm{app}}(y)$ involving the Hurwitz divisor and the monotonicity of the lower bound as $g \to \infty$, one can verify that $y = 0.19$ works for $g \geq 44$. An explicit computation for the remaining cases can be done to check that the lower bounds displayed in Figure 7 work for $13 \leq g \leq 43$ and $y > 0.91$ works for $g = 12$ (for $g = 12$ and $g = 14$, we use the divisor $\mathrm{NF}_{(2g-2)}$ instead of the Hurwitz divisor). ☐

We rewrite now the contributions of c_Γ and w_Γ^{mid} in terms of the parameters characterizing boundary divisors. We frequently drop the index Γ in the sequel to lighten the notation.

Lemma 8.5. *The contribution of the canonical bundle in terms of the parameters classifying boundary divisors is given by*

$$c_\Gamma \;=\; \Big(1-\frac{\kappa}{N}\Big)(M^\top + n^\top + P) - M^\top_{-1} - P_{-1} - \frac{\kappa}{N}\Big(v^\top + R\Big)\,. \tag{75}$$

Proof. Rewriting all objects in c_Γ in terms of the top level versions gives

$$c_\Gamma \;=\; \frac{\kappa}{N}N^\perp - \kappa^\perp - \frac{\kappa}{N}R \;=\; \kappa^\top - \frac{\kappa}{N}N^\top - \frac{\kappa}{N}R\,.$$

Substituting in this expression

$$\begin{aligned} \kappa^\top \;&=\; M^\top + n^\top - M^\top_{-1} + P - P_{-1}\,,\\ N^\top \;&=\; M^\top + n^\top + P + v^\top \end{aligned} \tag{76}$$

gives the claim. □

Using the previous lemmas, we write the full main estimate that we want to control as

$$\begin{aligned} &s_\Gamma(y) - 12y\frac{w_\Gamma^{\mathrm{app}}(\mu) - w_\Gamma^{\mathrm{mid}}(\mu)}{w_\lambda^{\mathrm{app}}(\mu)}\\ =\;& c_\Gamma + y\frac{12 w_\Gamma^{\mathrm{mid}}(\mu)}{w_\lambda^{\mathrm{app}}(\mu)} + (1-y)b_\Gamma\\ \ge\;& \Big(\frac{6}{w_\lambda^{\mathrm{app}}}y - \frac{\kappa}{N}\Big)(v^\top - 1) + (1-y)b_\Gamma - P_{-1} - \frac{\kappa}{N}R\\ +\;& \frac{1+M_{-1}}{N}(M^\top + n^\top + P) \;-\; M^\top_{-1} \;+\; y\cdot\frac{12}{w_\lambda^{\mathrm{app}}}\frac{\kappa^\perp}{\kappa} - \frac{\kappa}{N} \qquad =: T_1 + T_2 \end{aligned} \tag{77}$$

where the T_i terms on the right-hand side of the above expression are labeled one for each line.

We start by showing an estimate for T_1 in the case of a general stratum. From now on we fix

$$\varepsilon := \frac{11g-2}{4g^2+16g-8}\,. \tag{78}$$

Lemma 8.6. *If we have*

$$\frac{w_\lambda^{\mathrm{app}}}{6} \le y \le \frac{1}{4} - \varepsilon\,, \tag{79}$$

then $T_1 \ge -3P_{-1}^{\mathrm{RBT}}$ for g large enough. Unless Γ is a dumbbell graph with a rational bottom that carries all the marked points, we have the stronger estimate $T_1 \ge -2P_{-1}^{\mathrm{RBT}}$.

Proof. The lower bound is obtained by imposing the coefficient of $v^\top$ to be non-negative. By using $\kappa/N \le 1$, we see that if the lower bound holds then we have

$$\Big(\frac{6}{w_\lambda^{\mathrm{app}}}y - \frac{\kappa}{N}\Big)(v^\top - 1) \ge \Big(\frac{6}{w_\lambda^{\mathrm{app}}}y - 1\Big)(v^\top - 1) \ge 0\,.$$

In the case of odd genus g, we use the Brill–Noether divisor in the expression of b_Γ, while for even genus g, we need to use the Hurwitz divisor. Using the expression of the Hurwitz divisor obtained in Lemma 6.8, we can check that b_Γ is smaller in the Hurwitz case. More specifically, let us define

$$\delta^{\mathrm{BN}}_{\mathrm{Hur}} = \begin{cases} \frac{g^2-3g+2}{3g^3+32g^2+61g-24} & g \text{ even},\\ 0 & g \text{ odd}. \end{cases}$$

Using $\kappa/N < 1$ and the estimate (73) for Brill–Noether (or the analogous one for Hurwitz) we find, thanks to the lower bound for y, that

$$
\begin{aligned}
T_1 \geq & \left(\frac{1-4y}{2} - 2(1-y)\left(\frac{2}{g+3} + 5\delta^{\mathrm{BN}}_{\mathrm{Hur}}\right)\right) P^{\mathrm{NCT}}_{-1} \\
& + \left(9 - 12y - 2(1-y)\left(\frac{24}{g+3} + 60\delta^{\mathrm{BN}}_{\mathrm{Hur}}\right)\right) P^{\mathrm{OCT}}_{-1} \\
& + \left(7 - 12y - 2(1-y)\left(\frac{24}{g+3} + 60\delta^{\mathrm{BN}}_{\mathrm{Hur}}\right)\right) P^{\mathrm{EDB}}_{-1} \\
& - 2P^{\mathrm{RBT}}_{-1} - \frac{\delta^{\mathrm{H}}_{\Gamma}}{\ell_{\Gamma}}\,.
\end{aligned}
\tag{80}
$$

The upper bound for y in (79) is exactly the one that makes the coefficient of P^{NCT}_{-1} in the previous expression positive. Moreover it also implies that all the other terms in the brackets are positive for g large enough. This shows the first claim.

For the strengthening claim we observe that $\frac{\delta^{\mathrm{H}}_{\Gamma}}{\ell_{\Gamma}} \leq 1/p_e$ for every e. This ramification term is covered by the P^{EDB}_{-1}-summand or the P^{OCT}_{-1}-summand, if at least one such edge exists. The only ramified boundary divisors whose level graphs do not have such an edge are exactly dumbbell graphs with a rational bottom that carries all the marked points. Since for these special graphs Γ we have $\ell_{\Gamma} = P^{\mathrm{RBT}}_{-1}$, we have shown the full claim. □

Remark 8.7. There are sequences of connected strata for which it is impossible to show bigness of the canonical class or general type for all but finitely many cases by just using the Brill–Noether (and Hurwitz) divisors and the approximation to the mid-range Weierstrass divisor W^{app}.

Indeed if we impose no constraints on a sequence of signatures μ_k, there are graphs Γ_k giving a boundary divisor of $\mathbb{P}\mathrm{MS}(\mu_k)$ for which the term T_2 tends to negative infinity for k growing and the term T_1 stays bounded. Consider the example of $\mu_k = (k-2, 2^{k/2})$ and the case of a graph Γ where only the zero of high order $k-2$ is on bottom level and where P is independent of k. Then the only linear terms in k of T_2 are the positive term $(1+M_{-1})(M^{\top}+n^{\top})/N = k/10+O(1)$ and the negative term $-M^{\top}_{-1} = -k/6$, while all the other terms are bounded. Note also that if $v^{\top}$ is independent of k, then also T_1 is independent of k, so $s^{\mathrm{mid}}_{\Gamma}(y) < 0$ for any y for almost any k.

8.3. **The (non-minimal) strata with few zeros.** We exclude the minimal strata from this section, since one source of ramification divisors at the boundary, the HBB graphs, occurs only for the minimal strata and since we will make the bound effective for them.

We estimate the summands T_1 and T_2 of (77). Recall that by Lemma 7.8 and the subsequent remark $w^{\mathrm{app}}_{\lambda} \to 1/2$ as $g \to \infty$ for any sequence of strata with a uniformly bounded number of zeros and that $w^{\mathrm{app}}_{\lambda} = w^{\mathrm{mid}}_{\lambda}$ in case all the zeros are of even order.

We say that a stratum has *few zeros* if

$$
M_{-1} \leq \frac{12}{w^{\mathrm{app}}_{\lambda}} \frac{N}{\kappa} \left(\frac{1}{4} - \varepsilon\right) - 1 \tag{81}
$$

where ε was defined in (78). The above condition implies that $w^{\mathrm{app}}_{\lambda} \to 1/2$. Indeed, since by Remark 7.9 we know that $w^{\mathrm{app}}_{\lambda}$ is a bounded function of g, by (81) we also have that M_{-1} is a bounded function in the case of strata with few zeros. But this, again by Remark 7.9, implies that the limit for $g \to \infty$ of w_λ^{app} is equal to the limit of w_λ^{mid}, which is $1/2$. Hence, since $\kappa < N$, the condition 'few zeros' is implied by

$$M_{-1} \leq 5 - 24\varepsilon' \tag{82}$$

where ε' is a function going to zero for $g \to \infty$. In particular choosing $\varepsilon' < 11/48$ implies that strata with $n \leq 10$ qualify for 'strata with few zeros'. Obviously, the condition depends on the distribution of zero orders. For example, if the zero type μ does not have simple zeros, then strata with up to 15 higher order zeros qualify for 'strata with few zeros'.

Lemma 8.8. *Let $y = 1/4 - \varepsilon$ be as above. Then, for all but a finite number of strata with few zeros, the following estimates hold: If Γ is a dumbbell graph with a rational bottom vertex that carries all marked points, then $T_2 \geq 3P_{-1}^{\mathrm{RBT}}$. For all other Γ we have $T_2 \geq 2P_{-1}^{\mathrm{RBT}}$.*

Note that the condition 'few zeros' depends on ε, i.e., the smaller ε is chosen the larger we need to take g for both the T_1-estimate and the T_2-estimate to hold.

Proof. Using that

$$M^\top + n^\top + P \;=\; N - \kappa^\perp - 1 - M_{-1}^\perp + P_{-1}$$

we can rewrite the T_2 expression above as

$$T_2 \;=\; \left(y\frac{12}{w_\lambda^{\mathrm{app}}\kappa} - \frac{1+M_{-1}}{N}\right)\kappa^\perp + \left(1 - \frac{1+M_{-1}}{N}\right)M_{-1}^\perp + \frac{1+M_{-1}}{N}P_{-1}\,. \tag{83}$$

The condition on few zeros ensures that with $y = \frac{1}{4} - \varepsilon$ the first summand is positive for g large. In the absence of rational tails each of the terms is positive and we are done. We need to refine this in the presence of rational tails. Note that each of the quantities $\kappa^\perp$, $M^\perp$, $M_{-1}^\perp$ and P_{-1} can be interpreted as a sum over the vertices on bottom level. For each such vertex v we thus define accordingly P_v and $P_{-1,v}$ etc, and write $\kappa_v := \kappa_v^\perp$ or $M_v := M_v^\perp$ etc, since our focus is on bottom level anyway.

We will apply this mainly for v being a rational tail vertex. In this case $\kappa_v = n_v - 1 - M_{-1,v} + 1/p_e$ where e is the rational tail edge and $p_e = M_v + 1$. Using the trivial estimate for non-rational tails we deduce that

$$\begin{aligned} T_2 \geq \sum_{v \in V^{\mathrm{RBT}}} &\left(y\frac{12}{w_\lambda^{\mathrm{app}}\kappa} - \frac{1+M_{-1}}{N}\right)(n_v - 1) \\ &+ \left(1 - y\frac{12}{w_\lambda^{\mathrm{app}}\kappa}\right)M_{-1,v} + \frac{12y}{w_\lambda^{\mathrm{app}}\kappa}\frac{1}{M_v+1}\,. \end{aligned} \tag{84}$$

The first term is non-negative for every v and the last is positive, a negligibly small multiple of $1/p_e$. We use the middle summand to get the required positivity. For this we note that

$$M_{-1,v} \geq \frac{3}{M_v + 1} = \frac{3}{p_e} \tag{85}$$

with equality if and only if $n_v = 2$ and $M_v = 2$, i.e., for rational tails the three legs. Indeed, since the sum of reciprocals is minimized by the equidistributed situation, we have

$$M_{-1,v} \;=\; \sum_{i=1}^{n_v}\frac{1}{m_i+1} \geq \frac{n_v^2}{M_v + n_v}$$

and

$$\frac{n_v^2}{M_v+n_v}-\frac{3}{M_v+1} \;=\frac{(n_v^2-3)M_v+n_v(n_v-3)}{(M_v+n_v)(M_v+1)} \tag{86}$$

which is zero if $n_v = M_v = 2$ and positive otherwise, since $n_v \geq 2$ and since $M_v \geq n_v$.

Suppose that Γ is not a rational bottom dumbbell. Then we use (85) to get that

$$\left(1-\frac{1+M_{-1}}{N}\right)M_{-1,v}>\frac{2}{p_e} \tag{87}$$

for large g (since $\frac{1+M_{-1}}{N} \leq y\frac{12}{w_\lambda^{\mathrm{app}}\kappa} = O(1/g)$ by condition (81)). Summing these contributions gives the term $2P_{-1}^{\mathrm{RBT}}$ we wanted.

Finally consider rational bottom dumbbell graphs with all marked points on bottom level. We have to improve the above estimate by $1/p_e = 1/(2g-1)$. For these graphs with $M_v = 2g-2$, for any small $1 > \delta > 0$ we find the analogue of (86) in this situation to be

$$(1-\delta)M_{-1}-\frac{3}{p_e}\geq\frac{((1-\delta)n^2-3)(2g-2)+n((1-\delta)n-3)}{(2g-2+n)(2g-1)}\,.$$

The previous expression is positive for $n \geq 4$. For $n = 2, 3$, one can check that it is positive for $\delta = 1/8$ and g large enough. Since $\frac{1+M_{-1}}{N} \leq y\frac{12}{w_\lambda^{\mathrm{app}}\kappa} \leq 1/8$ for g large enough, we have proven the statement. □

8.4. **The equidistributed strata.** We consider now strata of type $\mu = (s^n)$. For these strata the parameters $M_\Gamma^\top$ and $n_\Gamma^\top$ are dependent, since $M_\Gamma^\top = s \cdot n_\Gamma^\top$. The main quantities for such strata in terms of s and n are

$$g \;=\; \frac{s}{2}n+1\,,\qquad N \;=\; (s+1)n+1\,,\qquad \kappa_\mu \;=\; n\frac{s(s+2)}{s+1}\,. \tag{88}$$

Moreover in this case

$$\frac{\kappa^\perp}{\kappa} \;=\; 1-\frac{n^\top}{n}+\frac{P_{-1}-P}{\kappa}.$$

We present now the analogue of Lemma 8.8 in the case of equidistributed strata.

Lemma 8.9. *For all but a finite number of equidistributed strata, there is some y satisfying condition (79) such that the following estimates hold: If Γ is a dumbbell graph with a rational bottom vertex that carries all marked points, then $T_2 \geq 3P_{-1}^{\mathrm{RBT}}$. For all other Γ we have $T_2 \geq 2P_{-1}^{\mathrm{RBT}}$.*

Proof. In the case of equidistributed strata, we can either consider the expression for T_2 given in (83) or an equivalent expression given by

$$\begin{aligned} T_2 \;=\; & \frac{ns(s+2)}{(s+1)((s+1)n+1)}\frac{n^\top}{n}+\frac{12}{w_\lambda^{\mathrm{mid}}}y\Big(1-\frac{n^\top}{n}\Big)-\frac{\kappa}{N} \\ & +\left(\frac{1+M_{-1}}{N}-y\frac{12}{w_\lambda^{\mathrm{mid}}\kappa}\right)P+y\frac{12}{w_\lambda^{\mathrm{mid}}\kappa}P_{-1}\,. \end{aligned} \tag{89}$$

Note that the coefficient of $\kappa^\perp$ in (83) is exactly the negative of the coefficient of P in the previous displayed equation. We first consider the range of parameters $n \leq 2(s+1)$ (we call this the range of few zeros) together with the choice $y = w_\lambda^{\mathrm{app}}/4$. With this choice the coefficient of $\kappa^\perp$ in (83) is positive. In fact, plugging in the quantities from (88) yields a rational function in (n, s) with positive denominator and a quadratic polynomial in n with $\mathbb{Q}[s]$-coefficients in the numerator with top coefficient $-s(2+s)$. It thus suffices to check the positivity at the boundary values $n=2$ and $n=2(s+1)$.

We then consider the complementary range of parameters $n>2(s+1)$ (we call this the range of many zeros) together with $y=w_\lambda^{\mathrm{app}}/6$. With this choice the coefficient of P in (89) is positive. In fact, plugging in yields a rational function, which when expressed in the shifted variables $s'=s-1\geq 0$ and $n'=n-2(s+1)\geq 0$ has exclusively non-negative coefficients.

In the range of few zeros, since it is clear that the expression (83) is positive, if there are no rational tails then we are done. In the range of many zeros and in absence of rational tails, we only need to show that the first line of (89) is positive. Once can check that the expression is minimized for the maximum value $n^\top=n-1$, and this bound already gives a positive expression for the first line of (89).

If there are rational tail edges we can consider the contribution of each rational tail edge separately, as we did in the proof of Lemma 8.8. Via a numerical check given by specializing (84) and using the fact that by stability every bottom level vertex of a rational tail has at least two legs, we can show that indeed in this case $T_2\geq 2P_{-1}^{\mathrm{RBT}}$. Similarly, we can also numerically check that in the case of a dumbbell with rational bottom and $n^\top=0$, we obtain the stronger bound $T_2\geq 3P_{-1}^{\mathrm{RBT}}$. $\square$

8.5. **The minimal strata with odd spin.** For the minimal strata there is no discussion of odd order zeros nor of rational tails, but we want to make the estimates effective. We give again estimates for the terms in (77).

Lemma 8.10. *Let $y=0.19$ and $g\geq 44$. Then we have $T_1>0$ for all graphs apart from a banana graph or a double banana graph with two vertices of genus one on top level, which are also HBB graphs, for which we have $T_1>-1/\ell_\Gamma$. Moreover*

$$T_2\geq 2\frac{g}{g-1}\left(1-\frac{P}{\kappa}\right). \tag{90}$$

Proof. Since one can check that, for $g\geq 44$, our choice of y satisfies the condition (79), we can argue as in the proof of Lemma 8.6 and effectively check that $T_1\geq 0$ apart from the case of an HBB.

In the case of an HBB, if there is a separating edge e, the additional term $-1/\ell_\Gamma$ is compensated by the $1/p_e$ contribution. Moreover, if there are at least three vertices, then the $v^\top$-term in T_1 is also enough to compensate the negative term coming from ramifications. The same is true if $v^\top=2$ and $\ell_\Gamma>1$. Hence the first part of the statement is proved.

In order to show the second part of the claim, it is enough to note that in the case of strata with $n^\top=0$, which is the case of the minimal strata, we can simply specialize T_2 and obtain the estimate

$$T_2\ \geq\ \frac{12}{w_\lambda^{\mathrm{mid}}}y\left(1-\frac{P-P_{-}}{\kappa}\right)+(1+M_{-1})\frac{P}{N}-\frac{\kappa}{N}\geq\left(\frac{12}{w_\lambda^{\mathrm{mid}}}y-\frac{\kappa}{N}\right)\left(1-\frac{P}{\kappa}\right).$$

A a numerical check shows that for $g\geq 44$ and $y=0.19$ the coefficient of the previous expression satisfies the desired bound. $\square$

8.6. **Proofs of Proposition 8.1 and Proposition 8.2.** We are now ready to prove the main result of this section. First we show the result for strata with few zeros and for equidistributed strata.

Proof of Proposition 8.1. In the case of strata with *few even order zeros*, we combine Lemma 8.6 and Lemma 8.8 to deduce from (77) that $s_\Gamma(1/4-\varepsilon) \geq T_1+T_2 \geq 0$ for almost all strata with few zeros.

For *equidistributed strata with s even*, we similarly combine Lemma 8.6 and Lemma 8.9 to obtain the result. Indeed note that by Remark 7.9, both $y = w_\lambda^{\mathrm{app}}/4$ in the range of few zeros and $y = w_\lambda^{\mathrm{app}}/6$ in the range of many zeros satisfy the condition (79).

The rest of the proof deals with the modification for strata with *odd entries*. In this case we want to improve the T_2-bounds of Lemma 8.8 and Lemma 8.9 by the absolute value of the lower bound

$$y\frac{12}{w_\lambda^{\mathrm{app}}}(w_\Gamma^{\mathrm{app}}(\mu) - w_\Gamma^{\mathrm{mid}}) \geq -y\frac{12}{8w_\lambda^{\mathrm{app}}}\left(M_{-1}^\perp - \frac{\kappa^\perp}{\kappa}M_{-1}\right) \tag{91}$$

coming from Lemma 7.8.

Consider the case of *equidistributed strata with s odd*. In order to strengthen Lemma 8.9, we first look at the situation without rational tail edges. In the range of few zeros, i.e., for $n \leq 2(s+1)$, with $y = w_\lambda^{\mathrm{app}}/4$ (so when the coefficient of $\kappa^\perp$ in (83) is positive), the negative term (91) is compensated by the coefficient $\left(1 - \frac{1+M_{-1}}{N}\right)$ of $M_{-1}^\perp$ in (83). Indeed $\frac{1+M_{-1}}{N}$ in the range of few zeros is $O(1/g)$, while $y\frac{12}{8w_\lambda^{\mathrm{app}}} = 3/8$. In the range of many zeros and for $y = w_\lambda^{\mathrm{app}}/6$, we can rewrite the right-hand side of the odd negative contribution (91) as

$$-\frac{1}{4}\left(M_{-1}^\perp - \frac{n^\perp}{n}M_{-1}\right) - \frac{P-P_-}{4\kappa}M_{-1}\,.$$

One can check that the first line of (89) compensates for the first negative term of the previous displayed expression, while the second line of (89) compensates for the second negative term. In the presence of rational tail edges, the proof for s even is a numerical check. Further more, one checks that the additional negative odd contribution (91) does not spoil the required positivity.

In the case of strata with few zeros which are of odd order, we concentrate only in the situation of strata with two odd zeros, i.e., $\mu = (m, 2g-2-m)$ with $m > 0$ odd.

In this case we can follow the same strategy that we used for strata with few zeros. We want to choose y such that the coefficient of $\kappa^\perp$ in (83) is positive, which means

$$y \geq \frac{\kappa}{N}(1+M_{-1})\frac{w_\lambda^{\mathrm{app}}}{12}\,.$$

We choose $y = 1/6$ (which also satisfies the condition imposed by Proposition 7.10 of $y > 1/24$), so that the previous condition is satisfied. Since for our choice of y the coefficient of $M_{-1}^\perp$ in (91) becomes $-1/(4w_\lambda^{\mathrm{app}})$ and $w_\lambda^{\mathrm{app}} \to 1/2$, we see that if there are no rational tails, the coefficient of $M_{-1}^\perp$ of (83) compensates this term and hence $T_1 + T_2 \geq 0$.

In the case of a dumbbell graph with a rational bottom vertex (and hence with both legs on bottom level), one can numerically check that the expression

$$T_2 - 3/P - \frac{1}{4w_\lambda^{\mathrm{app}}}M_{-1}\left(1 - \frac{\kappa^\perp}{\kappa}\right)$$

is positive. Indeed one can express the previous expression as a rational function depending on g and M_{-1} and use the bounds $1/2 + 1/(2g-2) \geq M_{-1} \geq 2/g$ in order to find an expression only depending on g, which is then easy to check to be positive for large g. $\square$

Now we show the effective estimate for the minimal strata.

Proof of Proposition 8.2. Thanks to Lemma 8.10, except for two special HBB cases, since $P/\kappa < 1$ we have $s_\Gamma(0.19) \geq T_1 + T_2 > 0$ for $g \geq 44$. For the two special HBB graphs, we only need to show that the bound of the T_2 term dominates the negative term $-1/\ell_\Gamma$ in the bound for the T_1 term.

In the cases of a double banana HBB graph with two vertices of genus one on top level, we have $P = 4$, hence from (90) we obtain $T_2 > 1$ for $g \geq 4$, which is enough to compensate the negative term $-1/\ell_\Gamma = -1$.

In the case of an HBB banana graph, we can consider the bound $-1/\ell_\Gamma \geq -2/P$ and write $P = 2g^\top$. Hence we need to check that the expression

$$T_1 + T_2 > -\frac{1}{g^\top} + 2\frac{g}{g-1}\left(1 - \frac{2g^\top}{\kappa}\right)$$

is positive. Using the bounds $1 \leq g^\top \leq g-1$, we can check that the expression is positive for $g \geq 44$.

The justification for the general type statement for $g < 44$ is assisted by computer programs and proved for the ranges of y in Figure 7. In this case we use the version of D_{NC} given by Proposition 5.12 and its refinement Proposition 5.13. We remark that for $g = 14$ we need to use the $\mathrm{NF}_{(2g-2)}$ divisor instead of the Hurwitz divisor Hur_μ. Moreover, for $g \leq 18$, we need to use the full shape of the more refined D_{NC} compensation divisor given by Proposition 5.13. In order to use this refinement, we need to be able to list all possible prong distributions on multi-banana graphs, which is not feasible for large g. Therefore, we use two different programs for $g \leq 18$ and $18 < g < 44$. Furthermore, it is still not feasible to list all two-level graphs in this range, hence we give some explanations on how to simplify the check.

First, for all the ranges of y in Figure 7, the compact type contribution of the Brill–Noether divisor or the Hurwitz divisor is larger than the non-compact type contribution, by an amount that beats the larger D_{NC}-correction for compact type, with two exceptions: EDB graphs, that are checked separately, and tails with elliptic top in $g = 13$, which require us to run an additional extra loop over these tails, that is performed on top of the below described procedure. This check about compact-type contributions can be done by comparing the P_{-1}^{NCT}-coefficient and the P_{-1}^{OCT}-coefficient appearing in (80), but we have to use a slightly different expression for them since we now have to use Proposition 5.13.

With this observation only the edges of Γ and their prongs are relevant, not the full graph structure. More precisely, in the minimal strata (where $M^\top = n^\top = 0$) the expression (77) depends on $v^\top$, P and P_{-1} only, where the latter two implicitly depend on $g^\top$ and E. For $g \leq 18$ the program checks positivity of (77) by a loop over $v^\top, g^\top, E$ and all possible prong distributions.

Second, in the range $18 < g < 44$, we only consider $v^\top = 1$. To justify this, we check that the coefficient of $(v^\top - 1)$ in (77) is positive so that we may drop this term. Now, given that this main estimate does no longer depend on $v^\top$, we may thus compare each graph with the graph where all top level vertices have been merged to one. This graph is being checked in our loop, and since we do not need to distinguish between compact type and non-compact type edges by the first observation, we obtain valid bounds by the merging procedure. In fact, this merging procedure might turn a graph of a ramified boundary divisor into an unramified case, but we can check that the positivity of the $(v^\top - 1)$-term outweighs this loss.

Finally, to avoid a loop over all partitions of P to cover all prong assignments to the edges, we instead make a case distinction on the sign of the P_{-1}-coefficient as a function of y. Depending on this sign, the interval of y that works for all the graphs with fixed $(E, g^\top)$ (and thus $P = 2g^\top - 2 + E$) is only constrained by the prong distribution that is either most equidistributed or most unbalanced. The computer program can thus be reduced to a simple loop over all possible $(E, g^\top)$, this case distinction on the P_{-1}-coefficient sign, and checking additionally the EDB graphs as well as the D_h-constraint.

The version of R given in Proposition 5.12 has a $v^\top$-term which is not present in the version given in (28) (which is used for the proof for $g \geq 44$). This term makes the lower bound of (79) bigger, since this bound is the one ensuring that the $v^\top$-coefficient is positive. As a result, the range given in Figure 7 for $g = 44$ does not include $y = 0.19$, that we proved to work for all $g \geq 44$. □

References

[AB21] D. Agostini and I. Barros. *Pencils on surfaces with normal crossings and the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$.* Forum Math. Sigma 9 (2021), Paper No. e31, 22.

[ACG11] E. Arbarello, M. Cornalba, and P. Griffiths. *Geometry of algebraic curves. Volume II.* Vol. 268. Grundlehren der Mathematischen Wissenschaften. With a contribution by J. Harris. Heidelberg: Springer, 2011, pp. xxx+963.

[Bar18] I. Barros. *Uniruledness of strata of holomorphic differentials in small genus.* Adv. Math. 333 (2018), pp. 670–693.

[BCGGM1] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller. *Compactification of strata of Abelian differentials.* Duke Math. J. 167.12 (2018), pp. 2347–2416.

[BCGGM2] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller. *The moduli space of multi-scale differentials.* (2019). arXiv: 1910.13492.

[BFV12] G. Bini, C. Fontanari, and F. Viviani. *On the birational geometry of the universal Picard variety.* Int. Math. Res. Not. IMRN 4 (2012), pp. 740–780.

[BIR08] M. Baker, S. Ih, and R. Rumely. *A finiteness property of torsion points.* Algebra Number Theory 2.2 (2008), pp. 217–248.

[BM21] I. Barros and S. Mullane. *Two moduli spaces of Calabi-Yau type.* Int. Math. Res. Not. IMRN 20 (2021), pp. 15833–15849.

[Boi15] C. Boissy. *Connected components of the strata of the moduli space of meromorphic differentials.* Comment. Math. Helv. 90.2 (2015), pp. 255–286.

[Bud21] A. Bud. *Maximal gonality on strata of differentials and uniruledness of strata in low genus.* Bull. Lond. Math. Soc. 53.6 (2021), pp. 1627–1635.

[Bul13] E. Bullock. *Subcanonical points on algebraic curves.* Trans. Amer. Math. Soc. 365.1 (2013), pp. 99–122.

[CC14] D. Chen and I. Coskun. *Extremal effective divisors on $\overline{M}_{1,n}$*. Math. Ann. 359.3-4 (2014), pp. 891–908.

[CG21] D. Chen and Q. Gendron. *Towards a classification of connected components of the strata of k-differentials*. (2021). arXiv: 2101.01650.

[Che17a] D. Chen. *Degenerations of Abelian differentials*. J. Differential Geom. 107.3 (2017), pp. 395–453.

[Che17b] D. Chen. *Teichmüller dynamics in the eyes of an algebraic geometer*. In: *Surveys on recent developments in algebraic geometry*. Vol. 95. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2017, pp. 171–197.

[Che19] D. Chen. *Tautological ring of strata of differentials*. Manuscripta Math. 158.3-4 (2019), pp. 345–351.

[CLS11] D.A. Cox, J.B. Little, and H.K. Schenck. *Toric varieties*. Vol. 124. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011, pp. xxiv+841.

[CMKV17] S. Casalaina-Martin, J. L. Kass, and F. Viviani. *The singularities and birational geometry of the compactified universal Jacobian*. Algebr. Geom. 4.3 (2017), pp. 353–393.

[CMSZ20] D. Chen, M. Möller, A. Sauvaget, and D. Zagier. *Masur-Veech volumes and intersection theory on moduli spaces of Abelian differentials*. Invent. Math. 222.1 (2020), pp. 283–373.

[CMZ20a] M. Costantini, M. Möller, and J. Zachhuber. *diffstrata – a Sage package for calculations in the tautological ring of the moduli space of Abelian differentials*. (2020). arXiv: 2006.12815.

[CMZ20b] M. Costantini, M. Möller, and J. Zachhuber. *The Chern classes and the Euler characteristic of the moduli spaces of abelian differentials*. (2020). arXiv: 2006.12803.

[CR86] M.C. Chang and Z. Ran. *The Kodaira dimension of the moduli space of curves of genus* 15. J. Differential Geom. 24.2 (1986), pp. 205–220.

[CR91] M.C. Chang and Z. Ran. *On the slope and Kodaira dimension of $\overline{M}_g$ for small g*. J. Differential Geom. 34.1 (1991), pp. 267–274.

[DM86] P. Deligne and G. D. Mostow. *Monodromy of hypergeometric functions and nonlattice integral monodromy*. Inst. Hautes Études Sci. Publ. Math. 63 (1986), pp. 5–89.

[EH16] D. Eisenbud and J. Harris. *3264 and all that—a second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016, pp. xiv+616.

[EH87] D. Eisenbud and J. Harris. *The Kodaira dimension of the moduli space of curves of genus* ≥ 23. Invent. Math. 90.2 (1987), pp. 359–387.

[EKZ14] A. Eskin, M. Kontsevich, and A. Zorich. *Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow*. Publ. Math. Inst. Hautes Études Sci. 120 (2014), pp. 207–333.

[Fa] *Flips and abundance for algebraic threefolds*. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992) (1992). Société Mathématique de France, Paris, 1992, pp. 1–258.

[Far09] G. Farkas. *Koszul divisors on moduli spaces of curves*. Amer. J. Math. 131.3 (2009), pp. 819–867.

[Far10] G. Farkas. *The birational type of the moduli space of even spin curves.* Adv. Math. 223.2 (2010), pp. 433–443.

[FJP20] G. Farkas, D. Jensen, and S. Payne. *The Kodaira dimensions of* $\overline{\mathcal{M}}_{22}$ *and* $\overline{\mathcal{M}}_{23}$. (2020). arXiv: `2005.00622`.

[FJP21] G. Farkas, D. Jensen, and S. Payne. *The non-abelian Brill-Noether divisor on* $\overline{\mathcal{M}}_{13}$ *and the Kodaira dimension of* $\overline{\mathcal{R}}_{13}$. (2021). arXiv: `2110.09553`.

[FL10] G. Farkas and K. Ludwig. *The Kodaira dimension of the moduli space of Prym varieties.* J. Eur. Math. Soc. (JEMS) 12.3 (2010), pp. 755–795.

[FM21] M. Fortuna and G. Mezzedimi. *The Kodaira dimension of some moduli spaces of elliptic K3 surfaces.* J. Lond. Math. Soc. (2) 104.1 (2021), pp. 269–294.

[FM22] G. Farkas and S. Mullane. *On the Kodaira dimension of Hurwitz spaces.* Math. Z. 300.4 (2022), pp. 3417–3432.

[For91] O. Forster. *Lectures on Riemann surfaces.* Vol. 81. Graduate Texts in Mathematics. Springer-Verlag, New York, 1991, pp. viii+254.

[FV13a] G. Farkas and A. Verra. *The classification of universal Jacobians over the moduli space of curves.* Comment. Math. Helv. 88.3 (2013), pp. 587–611.

[FV13b] G. Farkas and A. Verra. *The universal difference variety over* $\overline{\mathscr{M}}_g$. Rend. Circ. Mat. Palermo (2) 62.1 (2013), pp. 97–110.

[FV14] G. Farkas and A. Verra. *The geometry of the moduli space of odd spin curves.* Ann. of Math. (2) 180.3 (2014), pp. 927–970.

[Gee88] G. van der Geer. *Hilbert modular surfaces.* Vol. 16. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1988, pp. x+291.

[Gen18] Q. Gendron. *The Deligne-Mumford and the incidence variety compactifications of the strata of* ΩM_g. Ann. Inst. Fourier (Grenoble) 68.3 (2018), pp. 1169–1240.

[GHS07] V. A. Gritsenko, K. Hulek, and G. K. Sankaran. *The Kodaira dimension of the moduli of* $K3$ *surfaces.* Invent. Math. 169.3 (2007), pp. 519–567.

[Har77] R. Hartshorne. *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496.

[Har84] J. Harris. *On the Kodaira dimension of the moduli space of curves. II. The even-genus case.* Invent. Math. 75.3 (1984), pp. 437–466.

[HH09] B. Hassett and D. Hyeon. *Log canonical models for the moduli space of curves: the first divisorial contraction.* Trans. Amer. Math. Soc. 361.8 (2009), pp. 4471–4489.

[HM82] J. Harris and D. Mumford. *On the Kodaira dimension of the moduli space of curves.* Invent. Math. 67.1 (1982). With an appendix by William Fulton, pp. 23–88.

[KM98] J. Kollár and S. Mori. *Birational geometry of algebraic varieties.* Vol. 134. Cambridge Tracts in Mathematics. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. Cambridge University Press, Cambridge, 1998, pp. viii+254.

[Kon93] S. Kondō. *On the Kodaira dimension of the moduli space of $K3$ surfaces*. Compositio Math. 89.3 (1993), pp. 251–299.

[Kon99] S. Kondō. *On the Kodaira dimension of the moduli space of $K3$ surfaces. II*. Compositio Math. 116.2 (1999), pp. 111–117.

[KZ03] M. Kontsevich and A. Zorich. *Connected components of the moduli spaces of Abelian differentials with prescribed singularities*. Invent. Math. 153.3 (2003), pp. 631–678.

[Laz04] R. Lazarsfeld. *Positivity in algebraic geometry. I*. Vol. 48. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics. Classical setting: line bundles and linear series. Springer-Verlag, Berlin, 2004, pp. xviii+387.

[Li94] J. Li. *Kodaira dimension of moduli space of vector bundles on surfaces*. Invent. Math. 115.1 (1994), pp. 1–40.

[Log03] A. Logan. *The Kodaira dimension of moduli spaces of curves with marked points*. Amer. J. Math. 125.1 (2003), pp. 105–138.

[Mum83] D. Mumford. *Towards an enumerative geometry of the moduli space of curves*. In: *Arithmetic and geometry, Vol. II*. Vol. 36. Progr. Math. Birkhäuser Boston, Boston, MA, 1983, pp. 271–328.

[O'G89] K. G. O'Grady. *On the Kodaira dimension of moduli spaces of abelian surfaces*. Compositio Math. 72.2 (1989), pp. 121–163.

[Pet19] J. Petok. *Kodaira dimension of moduli of special $K3^{[2]}$-fourfolds of degree 2*. (2019). arXiv: `1911.03435`.

[Rei87] M. Reid. *Young person's guide to canonical singularities*. In: *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*. Vol. 46. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1987, pp. 345–414.

[Sch20] I. Schwarz. *On the Kodaira dimension of the moduli space of hyperelliptic curves with marked points*. (2020). arXiv: `2002.03417`.

[Sch21] I. Schwarz. *On the Kodaira dimension of the moduli space of nodal curves*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 22.1 (2021), pp. 463–484.

[Swa69] R. G. Swan. *Invariant rational functions and a problem of Steenrod*. Invent. Math. 7 (1969), pp. 148–158.

[Tai82] Y.S. Tai. *On the Kodaira dimension of the moduli space of abelian varieties*. Invent. Math. 68.3 (1982), pp. 425–439.

[TVA19] S. Tanimoto and A. Várilly-Alvarado. *Kodaira dimension of moduli of special cubic fourfolds*. J. Reine Angew. Math. 752 (2019), pp. 265–300.

[Wri15] A. Wright. *Translation surfaces and their orbit closures: an introduction for a broad audience*. EMS Surv. Math. Sci. 2.1 (2015), pp. 63–108.

[Zor06] A. Zorich. *Flat surfaces*. In: *Frontiers in number theory, physics, and geometry. I*. Springer, Berlin, 2006, pp. 437–583.

Department of Mathematics, Boston College, Chestnut Hill, MA 02467, USA
Email address: dawei.chen@bc.edu

Universität Duisburg-Essen Fakultät für Mathematik 45117 Essen, Germany
Email address: matteo.costantiuni-due.de

Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 6-8, 60325 Frankfurt am Main, Germany
Email address: moeller@math.uni-frankfurt.de