Hodge theory of Teichmüller curves

Martin Möller
Universität Duisburg-Essen

Mathematische Arbeitstagung June 10-16, 2005
Max-Planck-Institut für Mathematik, Bonn, Germany

Introduction

Let M_g be the moduli space of curves of genus $g \geq 2$ and let $\Omega M_g \to M_g$ be the total space of the Hodge bundle. There is a natural $\mathrm{GL}_2^+(\mathbb{R})$-action on the complement ΩM_g^* of the zero section in ΩM_g: For $A \in \mathrm{GL}_2^+(\mathbb{R})$ and $(X, \omega) \in \Omega M_g^*$ postcompose the charts given by integration of ω with the linear action of A on $\mathbb{C} \cong \mathbb{R}^2$. This defines a new complex structure on X and a one-from that is holomorphic for this complex structure. Moreover the action preserves the stratifiction of ΩM_g^* according to the number and multiplicities of the zeros of ω.

Some motivation why it seems worth studying this action:
First, the $\mathrm{GL}_2^+(\mathbb{R})$-action lifts in a natural way to the Hodge bundle over the Teichmüller space T_g. The images of orbits in T_g are complex geodesics for the Teichmüller (equivalently: Kobayashi-) metric.
In the -rare- case that the image C of such a geodesic in M_g is closed, C is called a *Teichmüller curve*. This happens if and only if the $\mathrm{GL}_2^+(\mathbb{R})$-orbit is closed in ΩM_g.
Second, the action sheds much light in the behaviour of trajectories on rational-angled billiard tables. See [MaTa] for a survey and the references therein.
Third, how do closures of $\mathrm{GL}_2^+(\mathbb{R})$-orbits look like ? M_g seems to behave as nicely as a symmetric domain with respect to this action. Compare the following two results:

Theorem 1 *(Ratner, special case) Let $M = \mathbb{H}^n/\Gamma$ be the quotient of hyperbolic n-space by a discrete group. Then the closure of an $\mathrm{SL}_2(\mathbb{R})$-orbit in the frame bundle $FM = \mathrm{SO}^0(1, n)/\Gamma$ is the orbit of a closed subgroup $H < \mathrm{SO}^0(1, n)$, whose intersection with some conjugate of Γ is a lattice.*

Theorem 2 *(McMullen, [Mc1]) In ΩM_2 an orbit closure is one of the following possibilities: i) the lift of a Teichmüller curve to ΩM_g; ii) the locus of eigenforms for real multiplication by a fixed order; iii) a stratum $\Omega M_2(2)$ or $\Omega M_2(1, 1)$.*

The arguments used to prove this are particular to genus 2. In higher genus the locus of eigenforms for real multiplication is no longer $\mathrm{GL}_2^+(\mathbb{R})$-invariant.

In the sequel we restrict ourselves to the case of closed orbits i.e. to Teichmüller curves and give a characterization valid in all genera as well as applications towards the classification of Teichmüller curves.

Teichmüller curves

Given a point $(X, \omega) \in \Omega M_g^*$, we denote by $\mathrm{Aff}^+(X, \omega)$ the group of orientation preserving diffeomorphism of X, that are affine with respect to the charts given by ω. There is a well-defined map $D : \mathrm{Aff}^+(X, \omega) \to \mathrm{SL}_2(\mathbb{R})$ by taking the matrix part of the diffeomorphism. Let Γ be the image of D. Then (X, ω) generates a Teichmüller curve if and only if Γ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. In this case, up to conjugation, $C = \mathbb{H}/\Gamma$.
This observation makes it possible to detect Teichmüller curves. Examples are torus coverings ramified over only one point, or regular $2n$-gons with opposite sides glued by translation. The latter belong to the first series of examples, discovered by Veech ([Ve]), where the trace field $K = \mathbb{Q}(\mathrm{tr}(\gamma), \gamma \in \Gamma)$ is different from $\mathbb{Q}$. K is always a number field. Let L be the Galois closure of $K/\mathbb{Q}$.

Theorem 3 *([Mo1]) If $C \to M_g$ is a Teichmüller curve, then there exists a finite unramified covering $C_1 \to C$, such that the variation of Hodge structures (VHS) of the family $f : \mathcal{X} \to C_1$ decomposes as follows:*

$$R^1 f_* L = (\oplus_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L/K)} \mathbb{L}^\sigma) \oplus \mathbb{M}.$$

Here $\mathbb{L}^{\mathrm{id}}$ is 'maximal Higgs', i.e. the corresponding representation $\pi_1(C) \to$ $\mathrm{Aut}(\text{Fibre of}\, \mathbb{L}^{\mathrm{id}})$ has image Γ. $\mathbb{M}$ splits off over $\mathbb{Q}$.
Conversely a family f whose VHS contains a maximal Higgs local subsytem of rank 2 defined over $\mathbb{R}$ comes from a finite unramified covering of a Teichmüller curve.

Hence Teichmüller curves 'behave' a little like Shimura curves, for which the VHS is built up only of local subsystems, that are maximal Higgs, and unitary local systems, see [ViZu] for more details.
Some consequences:

Corollary 4 *([Mo1]) The family of Jacobians over a Teichmüller curve contains a family of r-dimensional abelian subvarieties A_r with RM by K. Teichmüller curves are defined over number fields and the absolute Galois group of $\mathbb{Q}$ acts on the set of Teichmüller curves.*

Suppose (X, ω) generates a Teichmüller curve. A point P on X is called *periodic*, if the orbit $\mathrm{Aff}^+(X, \omega) \cdot P$ is finite. Examples of periodic points are the zeros of ω, preimages of torsion points for torus coverings and Weierstraß points for hyperelliptic Teichmüller curves. Periodic points give rise to a section of f for a suitable unramified covering $C_1 \to C$.

Theorem 5 *([Mo2]) For each unramified covering $C_1 \to C$ the Mordell-Weil group of the family of abelian varieties A_r/C_1 is finite. In particular if $r = g$ the difference of two periodic points is torsion.*

A Teichmüller curve generated by (X, ω) is called primitive, if (X, ω) does not arise via a covering from lower genus. If $r = g$, a Teichmüller curve is obviously primitive, but the converse does not hold.
Since there are 'few' torsion points lying on the image of a curve in its Jacobian, Theorem 5 'explains' why -contrary to the intuition coming from dimension counting- primitive Teichmüller curves in strata with many zeros are 'rare'. In fact the above result is used in [Mc2] to complete the classification of Teichmüller curves in genus 2.

References

[MaTa] Masur, H., Tabachnikov, S., *Rational billiards and flat structures*, Handbook of Dyn. Systems, Vol. 1 North-Holland (2002)

[Mc1] McMullen, C., *Dynamics of* $\mathrm{SL}_2(\mathbb{R})$ *over moduli space in genus two*, preprint (2003)

[Mc2] McMullen, C., *Teichmüller curves in genus two: Torsion divisors and ratios of sines*, preprint (2004)

[Mo1] Möller, M., *Variations of Hodge structures of Teichmüller curves*, preprint (2004)

[Mo2] Möller, M., *Periodic points on Veech surfaces and the Mordell-Weil group over a Teichmüller curve*, preprint (2004)

[Ve] Veech, W., *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. 97 (1989), 533–583

[ViZu] Viehweg, E., Zuo, K., *A characterization of Shimura curves in the moduli stack of abelian varieties*, J. of Diff. Geometry 66 (2004), 233–287