Goethe-Universität Frankfurt
Institut für Mathematik
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Algebra
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Übungsblatt 8

Aufgabe 1 (6 Punkte)

Let $(G, \circ)$ be a group and $(A, \cdot)$ an abelian group together with a group action

$$G \times A \to A, \qquad (\sigma, a) \mapsto \sigma(a).$$

by which is meant a group homomorphism $G \to \mathrm{Aut}(A)$ and where $\mathrm{Aut}(A)$ is the group of bijective group homomorphisms $A \to A$. Let $\mathrm{Map}(G, A)$ be the abelian group of maps $G \to A$. We define two subgroups of $\mathrm{Map}(G, A)$ as

$$Z^1(G, A) = \{f : f(\sigma \circ \sigma') = \sigma(f(\sigma')) \cdot f(\sigma) \text{ for all } \sigma, \sigma' \in G\},$$
$$B^1(G, A) = \{f : \text{there exists } a \in A \text{ such that} f(\sigma) = a \cdot \sigma(a)^{-1} \text{ for all } \sigma \in G\}.$$

(a) Show that $B^1(G, A) \subseteq Z^1(G, A)$.

The quotient group

$$H^1(G, A) := Z^1(G, A)/B^1(G, A)$$

is called first cohomology group of G in A. If L/K is a finite Galois extension, then the group $H^1(\mathrm{Gal}(L/K), L)$ is trivial.

(b) Use this to show that if L/K is cyclic and $\sigma \in \mathrm{Gal}(L/K)$ a generator, then the following conditions are equivalent for elements $b \in L$:

  (i) $\mathrm{Tr}_{L/K}(b) = 0$

  (ii) There exists an element $a \in L$ such that $b = a - \sigma(a)$.

Aufgabe 2 (4 Punkte)

Let $n > 1$. The dihedral group D_n (Diedergruppe) is the group of symmetries of the regular polygon with n vertices, which includes rotations and reflections.

(a) Compute the order of D_n

(b) Show that D_n is already generated by a reflection s and a rotation r

(c) Show that a p-Sylow subgroup of D_n is cyclic and normal for $p > 2$.

Aufgabe 3 (2 Punkte)

Let G be a finite p-group. Show that the number of non-normal subgroups of G is divisible by p.

Aufgabe 4 (4 Punkte)

(a) Show that a group of order 351 has a normal p-Sylow subgroup for some prime p dividing its order.

(b) Let b be a group with order pqr, where p, q, r primes such that $p < q < r$. Prove that G has a normal Sylow subgroup for either p, q or r.

Please, upload your solutions on the Olat page of this course, by **14:00** on **Tuesday, 12.01.2021**.