Goethe-Universität Frankfurt
Institut für Mathematik
Winter term 2020/21
13. Dezember 2020

Algebra
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Übungsblatt 7

Aufgabe 1 (4 Punkte)

(i) Show that $1+\sqrt[3]{2}$ is not a perfect square in $\mathbb{Q}(\sqrt[3]{2})$.

(ii) Show that $\sqrt[3]{3} \notin \mathbb{Q}(\sqrt[3]{2})$.

Aufgabe 2 (4 Punkte)

(i) Let K be a field and $L = K(a)$ a simple algebraic field extension with minimal polynomial $f_a \in K[X]$ of a. Show that $f(x) = \mathrm{N}_{L/K}(x-a)$ for all $x \in K$.

(ii) Show that if L/K is a finite non-separable field extension, then $\mathrm{Tr}_{L/K}(a) = 0$ for every $a \in L$.
Hints: either $L/K(a)$ or $K(a)/L$ is non-separable. Use Satz 2.34.

Aufgabe 3 (4 Punkte)

(i) We want to compute all *rational* points of the unit circle of equation $X^2+Y^2 = 1$ in $\mathbb{R}^2$. Prove that two rational numbers $a, b \in \mathbb{Q}$ satisfy $a^2+b^2 = 1$ if and only if there exist $m, n \in \mathbb{Z}$ such that

$$a = \frac{m^2-n^2}{m^2+n^2}, \qquad b = \frac{2mn}{m^2+n^2}.$$

Hint: use Hilbert Theorem 90 applied to the extension $\mathbb{Q}(i)/\mathbb{Q}$.

(ii) Let D be a positive square-free integer. Compute all *rational* points of the ellipse of equation $X^2 + DY^2 = 1$ in $\mathbb{R}^2$.

Aufgabe 4 (4 Punkte)

(i) Let E/K and E'/K be field extensions, where E'/K is finite Galois. Prove that:

(a) $E \cdot E'/E$ is finite Galois.

(b) The map $\varphi\colon \mathrm{Gal}(E\cdot E'/E) \to \mathrm{Gal}(E'/E\cap E')$, $\sigma \mapsto \sigma|_{E'}$ is a group isomorphism.

(c) If also E/K is finite, then $[E\cdot E' : K] = \frac{[E:K][E':K]}{[E\cap E':K]}$.

Hint: this is a refinement of Übungsblatt 5, Aufgabe 1.

(ii) Let L be the splitting field of $f(X) := X^7 - 7 \in \mathbb{Q}[X]$, let ζ be a 7-th primitive root of unity, and let $\alpha := \sqrt[7]{7}$. Compute $[L:\mathbb{Q}]$, prove that $\mathrm{Gal}\big(L/\mathbb{Q}(\alpha)\big)$ and $\mathrm{Gal}\big(L/\mathbb{Q}(\zeta)\big)$ are cyclic, and compute their orders.

Please, upload your solutions on the Olat page of this course, by **14:00** on **Tuesday, 05.01.2021**.