Goethe-Universität Frankfurt
Institut für Mathematik
Winter term 2020/21
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Algebra
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Übungsblatt 11

Aufgabe 1 (3 Punkte)

Let $f \in K[X]$ be a separable polynomial of degree n, with coefficients in a field K of characteristic different from 2. Prove that the embedding of the Galois group of f over K into S_n, as permutations of the roots of f, has image in A_n if and only if the discriminant of f is a square in K.

Aufgabe 2 (5 Punkte)

Determine whether the following equations are solvable by radicals.

(i) $X^5 - 2X + 1 = 0$ over $\mathbb{Q}$.

(ii) $X^7 - 7X^6 + 13X^5 + X^4 - 27X^3 + 25X^2 - 3X - 1 = 0$ over $\mathbb{Q}$.

(iii) $X^7 + 4X^5 - \frac{10}{11}X^3 - 4X + \frac{2}{11} = 0$ over $\mathbb{Q}$.

(iv) $X^5 - X - 1 = 0$ over $\mathbb{Q}$.
Hints: the associated Galois group contains a permutation of the roots with cycle type $(2, 3)$, that is, a 2-cycle composed with a 3-cycle. (This follows from a reduction theorem due to Dedekind.) Also, there is no subgroup of S_5 of cardinality 30.

General hint: plot the graph of the polynomials using e.g. GeoGebra.

Aufgabe 3 (6 Punkte)

Let $\{0, 1\} \subseteq M \subseteq \mathbb{C}$. We denote by $\mathfrak{K}(M)$ the set of points in $\mathbb{C}$ that can be obtained by compass and straightedge constructions from M. Note that $F_{\sqrt{}} = \mathfrak{K}(\{0, 1\})$.

(i) Prove that $\mathfrak{K}(M)$ is a field.

(ii) Prove that if $z \in \mathfrak{K}(M)$, then z is contained in a Galois extension L of $\mathbb{Q}(M \cup \overline{M})$, where $\overline{M} = \{\overline{m} : m \in M\}$, whose degree is a power of 2.
Hint: Generalizing Proposition 5.11 of the skript, starting from M instead of $\{0, 1\}$, it is possible to prove that $z \in \mathfrak{K}(M)$ if and only if there exists a chain of field extensions
$$\mathbb{Q}(M \cup \overline{M}) = L_0 \subset L_1 \subset \cdots \subset L_n \subset \mathbb{C},$$
such that $z \in L_n$ and $[L_{i+1} : L_i] = 2$ for $i = 0, \ldots, n-1$. You can assume this.

(iii) Prove that $\mathfrak{K}(M)$ is an algebraic extension of $\mathbb{Q}(M \cup \overline{M})$ and that the degree over $\mathbb{Q}(M \cup \overline{M})$ of any element $z \in \mathfrak{K}(M)$ is a power of 2.

Aufgabe 4 (3 Punkte)

Prove that the *angle trisection* is a geometric problem that can not be solved with compass and straightedge.
Hint: start with $M = \{0, 1, \zeta\}$, for some root of unity ζ.

Please, upload your solutions on the Olat page of this course, by **14:00** on **Tuesday, 02.02.2021**.