Goethe-Universität Frankfurt
Institut für Mathematik
Winter term 2020/21
2. November 2020

Algebra
Prof. Dr. Martin Möller
M.Sc. Riccardo Zuffetti

Übungsblatt 1

WARM-UP EXERCISES

Aufgabe 1 (3 Punkte)

(i) Let G be a group. Prove that $T := \{\alpha \in \mathrm{Aut}(G) \mid \alpha(U) = U \text{ for all subgroups } U \text{ of } G\}$ is a normal subgroup of $\mathrm{Aut}(G)$.

(ii) Prove that
$$G := \left\{ \left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right) \mid a, b, c \in \mathbb{Z}/3\mathbb{Z} \right\},$$
endowed with the usual matrix multiplication, is a group of order 27, where every element different from the identity matrix has order 3.
Use this to find two non-isomorphic finite groups for which for every n the number of elements of order n coincide for the two groups.

Aufgabe 2 (5 Punkte)

Let R be a ring. We denote by 1 the unit of R. ***Prove*** or ***disprove*** the following claims.

(i) If $x^2 = x$ for every $x \in R$, then R is commutative.

(ii) If I is an ideal of R such that $1 \in I$, then $I = R$.

(iii) If I is an ideal of R, then also $r(I) := \{x \in R \mid xa = 0 \text{ for all } a \in I\}$ is an ideal of R.

(iv) Let I and J be ideals of R, then also $\mathrm{prod}(I, J) := \{ab \mid a \in I \text{ and } b \in J\}$ is an ideal of R.

(v) If R is a PID (i.e. "Principal Ideal Domain", in German "Hauptidealring"), then the previous point is correct.

(vi) Consider $\mathbb{Z}[\sqrt{-5}] := \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$ as a subring of $\mathbb{C}$, where the operations are the ones inherited from the field $\mathbb{C}$. The map
$$N \colon \mathbb{Z}[\sqrt{-5}] \longrightarrow \mathbb{Z}, \qquad a + b\sqrt{-5} \longmapsto a^2 + 5b^2$$
is a ring homomorphism.

(vii) $\mathbb{Z}[\sqrt{-5}]$ is a PID.

Aufgabe 3 (2 Punkte)

Let R be a ring and let I, J be ideals of R.

(i) Prove that $I + J := \{a + b \mid a \in I \text{ and } b \in J\}$ is an ideal of R.

(ii) Define IJ the set of all elements of R that can be written as finite sums of elements of the form ab, where $a \in I$ and $b \in J$. Prove that IJ is an ideal of R.

Aufgabe 4 (6 Punkte)

(i) ***Prove*** or ***disprove*** the following claim.

The set of 2×2 *matrices of the form* $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$, *where* $a, b \in \mathbb{R}$, *endowed with the usual matrix operations, is a field.*

(ii) Prove that the following sets of real numbers are subfields of $\mathbb{R}$.

$$\mathbb{Q}[\sqrt{2}] \coloneqq \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$$
$$\mathbb{Q}[\sqrt[3]{2}] \coloneqq \{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Q}\}$$

Please, upload your solutions on the Olat page of this course, by **14:00** on **Tuesday, 10.11.2020**.