Goethe-Universität Frankfurt
Institut für Mathematik
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Algebraische Geometrie II
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Übungsblatt 3

Aufgabe 1 (4 Punkte)

Let (M,g) be a Riemannian manifold. Let $D : \mathcal{A}^0(TM) \to \mathcal{A}^1(TM)$ be a connection on the real tangent bundle TM. Let $D_v(u) \in TM$ be the tangent vector obtained by applying $D(v) \in \mathcal{A}^1(TM) \cong \mathcal{A}^0(\mathrm{End}(TM))$ to $u \in TM$. Recall that the Lie bracket $[\cdot,\cdot]\colon \mathcal{A}^0(TM) \times \mathcal{A}^0(TM) \to \mathcal{A}^0(TM)$ is defined locally by

$$[u,v] = \sum_j (\mathrm{d}b_j(u) - \mathrm{d}a_j(v))\frac{\partial}{\partial x_j}$$

for $u = \sum_i a_i \frac{\partial}{\partial x_i}$ and $v = \sum_i b_i \frac{\partial}{\partial x_i}$. Define the *torsion* of a connection D as

$$T_D : TM \times TM \to TM, \quad T_D(u,v) = D_u v - D_v u - [u,v].$$

(a) Show that $T_D \in \mathcal{A}^2(TM)$, namely that it is skew-symmetric and $\mathcal{A}^0$-linear.

(b) Write locally $D = \mathrm{d} + A$, where A is a one form with values in $\mathrm{End}(TM)$. Show that

$$T_D(u,v) = A(u)(v) - A(v)u$$

for any $u, v \in \mathcal{A}^0(TM)$.

Aufgabe 2 (4 Punkte)

Let (X,h) be a hermitian complex manifold. Let $g := \mathrm{Re}(h)$ be the induced Riemannian metric on the real tangent bundle.

(a) Show that, under the natural isomorphism $TX \cong T^{1,0}(X)$, any hermitian connection ∇_h on $T^{1,0}(X)$ induces a metric connection D_h on the Riemannian manifold (X,g), i.e. $\mathrm{d}g(u,v) = g(Du,v) + g(u,Dv)$.

(b) Let ∇_h be a hermitian connection with $T_{D_h} = 0$. Show that ∇_h is the Chern connection and that $\mathrm{Im}(h)$ is a Kähler form on X.

Remark: A metric connection D with $T_D = 0$ is called Levi-Civita connection. It is unique. One can prove also a converse of the last statement, so remember that if X is a Kähler manifold, the unique Chern connection on the holomorphic tangent bundle corresponds to the unique Levi-Civita connection on the real tangent bundle.

Aufgabe 3 (2 Punkte)

Let X be a Riemann surface. Show that for any line bundle $\mathcal{O}_X(D)$ it holds

$$c_1(\mathcal{O}_X(D)) = \deg(D).$$

Aufgabe 4 (6 Punkte)

Let $f : \mathcal{X} \to B$ be a family of Riemann surfaces. Recall that the flat vector bundle $\mathcal{H}$ over B given fiberwise by $\mathcal{H}_b = H^1(\mathcal{X}_b, \mathbb{C})$ defines a representation

$$\rho_f : \pi_1(B, b) \to \mathrm{Aut}(H^1(\mathcal{X}_b, \mathbb{C}))$$

for any $b \in B$. Compute the representation ρ_f for the two following families:

(a) $\chi = Z(y^2 = (x^2 - b)(x - 1))$, $B = \mathbb{D}_1(0) \setminus \{0\}$ is the punctured unit disk and the family map is $f : (x, y, b) \mapsto b$.

(b) $\chi = Z(y^2 = x(x - 1)(x - b))$, $B = \mathbb{A}^1_{\mathbb{C}} \setminus \{0, 1\}$ and the family map is $f : (x, y, b) \mapsto b$.

Hint: First of all compute the genus of $\mathcal{X}_b$. Then, for all $b \in B$, consider $\mathcal{X}_b$ as a ramified cover of $\mathbb{P}^1$ via the projection π_x and find a symplectic basis of $H_1(\mathcal{X}_b, \mathbb{C})$ by lifting loops in $\mathbb{P}^1 \setminus r_{\pi_x}$ via π_x, where r_{π_x} is the set of ramification points.

Abgabe: Zu Beginn der Übung um **14:15** Uhr am **Mittwoch, den 5. Juni**.