Goethe-Universität Frankfurt
Institut für Mathematik
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Algebraische Geometrie II
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Übungsblatt 2

Aufgabe 1 (4 Punkte)

(a) Prove that the singular cohomology groups of complex projective space are given by

$$\mathrm{H}^k(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Hinweis: If you are familiar with cellular homology, compute the CW-decomposition of $\mathbb{P}^n_{\mathbb{C}}$ and its homology.
Otherwise, use the Mayer–Vietoris long exact sequence

$$\cdots \to \mathrm{H}_{k+1}(\mathbb{P}^n_{\mathbb{C}}) \to \mathrm{H}_k(A \cap B) \to \mathrm{H}_k(A) \oplus \mathrm{H}_k(B) \to \mathrm{H}_k(\mathbb{P}^n_{\mathbb{C}}) \to \cdots$$

with $A = \mathbb{P}^n_{\mathbb{C}} \setminus \{z_n = 0\}$ and $B = \mathbb{P}^n_{\mathbb{C}} \setminus \{[0 : \cdots : 0 : 1]\}$.

(b) Compute the Hodge diamond of $\mathbb{P}^n_{\mathbb{C}}$.

(c) Prove that $\mathrm{Pic}(\mathbb{P}^n_{\mathbb{C}}) \cong \mathbb{Z}$ and that is generated by the divisor class of a hyperplane. *Hinweis: exponential sequence.*

Aufgabe 2 (4 Punkte)

Let $Y \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a codimension one subvariety. Prove that Y is given by the zero locus of a homogenous polynomial of degree d, for some $d \in \mathbb{N}$, and that Y is cohomologous to the union of d hyperplanes.

Aufgabe 3 (4 Punkte)

Prove that any holomorphic automorphism φ of $\mathbb{P}^n_{\mathbb{C}}$ is induced by a projective linear transformation $A_\varphi \in \mathrm{PGL}_{n+1}(\mathbb{C})$ of $\mathbb{C}^{n+1}$.

Hinweis: Consider the meromorphic function $\phi^*(x_i)$, where x_i is a standard coordinate of $\mathbb{P}^n_{\mathbb{C}}$, and its associated divisor.

Aufgabe 4 (4 Punkte)

Let X one of the complex manifolds below equipped with a hermitian metric h_X:

(a) $X = \mathbb{C}^n$ and h_X the standard hermitian metric.

(b) $X = \mathbb{P}^1_{\mathbb{C}}$ and h_X be the Fubini-Study metric.

(c) $X = \mathbb{H} = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$ together with the hyperbolic metric $h_X = \frac{\mathrm{d}z \otimes \mathrm{d}\overline{z}}{y^2}$.

Compute the curvature $F_{\nabla_h} : \mathcal{T}X \to \mathcal{T}X \otimes \mathcal{A}^2_X$ of the associated metric connections ∇_h, where $\mathcal{T}X$ is the holomorphic tangent bundle of X.

Moreover consider the fundamental $(1,1)$-form ω given $\omega = -\operatorname{Im}(h_X)$. Compare this fundamental form to the $(1,1)$-form induced by F_{∇_h}.

Identify the 2-sphere $\mathbb{S}^2 = \{(x_1, x_2, x_3) \colon x_1^2 + x_2^2 + x_3^2 = 1\} \subseteq \mathbb{R}^3$ with $\mathbb{P}^1_{\mathbb{C}}$ using the stereographic projection

$$\mathbb{S}^2 \to \mathbb{P}^1_{\mathbb{C}}, \quad (x_1, x_2, x_3) \mapsto [x_1 + ix_2 : 1 - x_3].$$

Check that under this identification the Riemannian metric on $\mathbb{P}^1_{\mathbb{C}}$ induced by the Fubini-Study metric is the same, up to constant, as the Riemannian metric induced by the standard metric of $\mathbb{R}^2$ on $\mathbb{S}^2$.

Abgabe: Zu Beginn der Übung um **14:15** Uhr am **Mittwoch, den 22. Mai**.