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Komplexe Algebraische Geometrie
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Übungsblatt 14

Aufgabe 1 (4 Punkte)

Let X be a complex torus, $\mu : X \times X \to X$ the addition map and $\pi : X \times X \to X$, $i = 1, 2$, the natural projections. Show that a C^∞-1-form ω on X is translation-invariant if and only if $\mu^*\omega = p_1^*\omega + p_2^*\omega$.

Aufgabe 2 (6 Punkte)

Let Γ be a free abelian group of finite rank. A Hodge-structure of weight 1 on Γ is a decomposition $\Gamma \otimes \mathbb{C} = H^{0,1} \oplus H^{1,0}$, where $H^{0,1}$ and $H^{1,0}$ are complex subvector spaces of $\Gamma \otimes \mathbb{C}$ with $H^{1,0} = H^{0,1}$. Show that giving a Hodge structure of weight 1 on Γ is equivalent to giving a complex structure on the real torus $(\Gamma \otimes \mathbb{C})/\Gamma$, i.e. an isomorphism of real tori $(\Gamma \otimes \mathbb{R})/\Gamma \to X$, where X is a complex torus.

Aufgabe 3 (2 Punkte)

Let $f : X \to Y$ be a holomorphic map between complex manifolds, where $n := \dim(X)$ and $m := \dim(Y)$. Let (U, φ) and (V, ψ) be local charts for X and Y respectively, such that $f(U) \subseteq V$.
Define $F = (F_1, \ldots, F_m) := \psi \circ f \circ \varphi^{-1}$. If f has rank k on U, i.e. the complex matrix $(\partial F_m/\partial z_k)_{m,k}$ has constant rank k in any point in $\varphi(U)$, then for every $x \in U$ there exist local charts (U', φ') and (V', ψ') of X and Y respectively such that $x \in U' \subseteq U$, $\varphi(U') \subseteq V'$, $\varphi(x) = 0$, $\psi(x) = 0$ and:

$$F' := \psi' \circ f \circ \varphi'^{-1} : (z_1, \ldots, z_n) \longmapsto (z_1, \ldots, z_k, 0, \ldots, 0).$$

(a) Use the previous result to prove the following:
Let f be as above, with $n > m$. Suppose that $y \in f(X)$ is such that the rank of f is maximal (i.e. $k = m$) on $f^{-1}(y)$. Then $f^{-1}(y)$ is a submanifold of X with dimension $n - m$.

(b) Show that

$$Z_t := \{z = (z_0 : \cdots : z_n) \in \mathbb{P}^n \mid z_0^t + \cdots + z_n^t = 0\}$$

is a compact submanifold of $\mathbb{P}^n$ for all $t \in \mathbb{N}_{>0}$. This is called *Fermat hypersurface*.

Aufgabe 4 (3 Punkte)

Let G be the complex Lie group

$$G := \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{C}) \right\},$$

biholomorphic, as a manifold, to $\mathbb{C}^3$. The group G (and every subgroup of it) acts on G by multiplication in $GL(3,\mathbb{C})$. Consider the group $\Gamma := G \cap GL(3, \mathbb{Z}+i\mathbb{Z})$. Then $(w_1, w_2, w_3) \in \Gamma$ acts on G by

$$(z_1, z_2, z_3) \longmapsto (z_1 + w_1, z_2 + w_1 z_3 + w_2, z_3 + w_3).$$

(a) Show that the quotient $X := G/\Gamma$ is a complex manifold of dimension three. This is called *Iwasawa manifold*.

(b) Show that the Iwasawa manifold is parallelizable.

Aufgabe 5 (4 Punkte)

Let E, F be two vector bundle on a complex manifold X, of rank e and f respectively. Let $\Phi : E \to F$ be a homomorphism of vector bundles such that $\operatorname{rank}(\Phi_x) = r\ (\leq e, f)$ for every $x \in X$.

(a) Show that exist a cover $X = \bigcup_i V_i$ of X and local trivializations (V_i, ψ_i) and (V_i, ψ_i') of E and F such that

$$\begin{aligned} \psi_i' \circ \Phi \circ \psi_i^{-1} : V_i \times \mathbb{C}^e &\to V_i \times \mathbb{C}^f \\ \big(x, (v_1, \ldots, v_e)\big) &\mapsto \big(x, (v_1, \ldots, v_r, 0, \ldots, 0)\big). \end{aligned}$$

(b) Use this result to describe the cocycles of the vector bundles $\ker(\Phi)$ and $\operatorname{coker}(\Phi)$.

Aufgabe 6 (2 Punkte)

Show that $\mathcal{O}(-1)\backslash s(\mathbb{P}^n)$ is naturally identified with $\mathbb{C}^{n+1}\backslash\{0\}$, where $s : \mathbb{P}^n \to \mathcal{O}(-1)$ is the zero-section. Use this to construct a submersion $S^{2n+1} \to \mathbb{P}^n$ with fiber S^1 (for $n = 1$ this yields the so called *Hopf fibration* $S^3 \to S^2$).