Goethe-Universität Frankfurt
Institut für Mathematik
Wintersemester 2018
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Komplexe Algebraische Geometrie
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Übungsblatt 10

Aufgabe 1 (4 Punkte)

Let M be a differential manifold and (U, φ) a local chart. We write φ in local coordinates as $\varphi = (x_1, \ldots, x_n)$, where $n = \dim(M)$. Let $\mathrm{d} : \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U)$ be maps such that:

- d is linear;
- $\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$ for all $f \in \mathcal{C}^\infty(U)$;
- $\mathrm{d}(\omega \wedge \alpha) = \mathrm{d}\omega \wedge \alpha + (-1)^k \omega \wedge \mathrm{d}\alpha$ if $\omega \in \mathcal{A}^k(U)$;
- $\mathrm{d}^2 f = 0$ for all $f \in \mathcal{C}^\infty(U)$.

(a) Prove that d is well-defined and that $\mathrm{d}^2(\omega) = 0$ also for every differential form ω.

(b) Prove that the map $\mathrm{d} : \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U)$ satisfying the previous property is unique for all k.

(c) The definition of d is local depending on U. Prove that all the d glue together to a well-defined map, called *exterior derivative*,

$$\mathrm{d} : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M).$$

(d) ★[1] Show that the pullback of a smooth map between differential manifolds induces a map of de Rham complexes (i.e. the pullback of forms commutes with d).

Aufgabe 2 (4 Punkte)

Let V be a finite dimentional vector space. Given $v \in V$, define for every $k \in \mathbb{N}$ the maps

$$i_v : \bigwedge^k V^* \longrightarrow \bigwedge^{k-1} V^*$$
$$\omega \longmapsto i_v(\omega)$$

where $i_v(\omega)(v_1, \ldots, v_{k-1}) = \omega(v, v_1, \ldots, v_{k-1})$ for every $v_1 \ldots, v_{k-1} \in V$.

(a) Check that $i_v(\omega \wedge \eta) = i_v(\omega) \wedge \eta + (-1)^k \omega \wedge i_v(\eta)$ if $\omega \in \bigwedge^k V^*$.
Hint: recall that if $\omega \in \bigwedge^k V^*$ and $\eta \in \bigwedge^\ell V^*$ then

$$\omega \wedge \eta(v_1, \ldots, v_{k+\ell}) = \sum_{\sigma \in S_{k,\ell}} \mathrm{sgn}(\sigma) \cdot \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot \eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})$$

for every $v_1, \ldots, v_{k+\ell} \in V$, where $S_{k,\ell}$ is the subset of the symmetric group $S_{k+\ell}$ containing all the permutations such that

$$\sigma(1) < \cdots < \sigma(k) \quad \text{and} \quad \sigma(k+1) < \cdots < \sigma(k+\ell).$$

[1] The exercises with ★ are optional.

(b) Let M be a differential manifold and let $X : U \to TU$ be a (local) vector field on an open $U \subseteq M$. We define $\mathrm{Lie}_X := \mathrm{d} \circ i_X + i_X \circ \mathrm{d}$ pointwise. Fix a local chart $\big(U, \varphi = (x_1, \dots, x_n)\big)$ and check that $\mathrm{Lie}_{\frac{\partial}{\partial x_j}}(f \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_r) = \frac{\partial f}{\partial x_j} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_r$, where $r \leq n$ and $f \in \mathcal{C}^\infty(U)$.

Aufgabe 3 (8 Punkte)

The aim of this exercise is to compute the de Rham cohomology in some examples. Recall that $H^0_{\mathrm{DR}}(\mathbb{R}^n) \cong \mathbb{R}$ and $H^q_{\mathrm{DR}}(\mathbb{R}^n) \cong 0$ when $q \geq 1$.

(a) Let $M = \coprod_{j=1}^n U_j$ where the U_j are connected open subsets of $\mathbb{R}^n$. What can you say about $H^q_{\mathrm{DR}}(M)$? Compute $H^0_{\mathrm{DR}}(M)$.

(b) Show that $H^q_{\mathrm{DR}}(S^1) \cong \begin{cases} \mathbb{R}, & \text{if } q = 0, 1, \\ 0, & \text{if } q \geq 2. \end{cases}$
Hint: use the Mayer-Vietoris sequence.

(c) Generalize the previous point showing that $H^q_{\mathrm{DR}}(S^n) \cong \begin{cases} \mathbb{R}, & \text{if } q = 0, n, \\ 0, & \text{otherwise.} \end{cases}$

(d) Show that $H^q_{\mathrm{DR}}(\mathbb{R}^2 \backslash \{2 \text{ pts}\}) \cong \begin{cases} \mathbb{R}, & \text{if } q = 0, \\ \mathbb{R}^2, & \text{if } q = 1, \\ 0, & \text{otherwise.} \end{cases}$
Hint: use Mayer-Vietoris.

(e) ★ Compute the de Rham cohomology of a cylinder, i.e. of $S^1 \times \mathbb{R}$.

(f) ★ Let $T = S^1 \times S^1$ be a torus. Show that $H^q_{\mathrm{DR}}(T) \cong \begin{cases} \mathbb{R}, & \text{if } q = 0, 2, \\ \mathbb{R}^2, & \text{if } q = 1, \\ 0, & \text{otherwise.} \end{cases}$
Hint: the easiest way is to compute directly the cohomology with the following result, called *Künneth theorem.*
Let M, N be differential manifolds and suppose that the cohomology of N is finite dimensional. Then for $0 \leq q \leq \dim(M) + \dim(N)$

$$H^q(M \times N) = \bigoplus_{a+b=q} H^a(M) \otimes H^b(N).$$

Try to solve this problem also without the Künneth formula, using again Mayer-Vietoris.

Abgabe Zu Beginn der Vorlesung um **10:00** am **Montag, den 21. Januar**.