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Mathematische Zeitschrift

ISSN 0025-5874
Volume 271
Combined 3-4

Math. Z. (2012) 271:757-779
DOI 10.1007/s00209-011-0889-4

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Received: 13 April 2010 / Accepted: 21 March 2011 / Published online: 13 May 2011

Abstract A compact Riemann surface of genus $g > 1$ has different uniform dessins d'enfants of the same type if and only if its surface group S is contained in different conjugate Fuchsian triangle groups Δ and $\alpha\Delta\alpha^{-1}$. Tools and results in the study of these conjugates depend on whether Δ is an arithmetic triangle group or not. In the case when Δ is not arithmetic the possible conjugators are rare and easy to classify. In the arithmetic case, i.e. for Shimura curves, the problem is much more complicated, but the arithmetic of quaternion algebras controls the growth of the number of uniform dessins of given type with respect to the genus. This number grows at most as $O(g^{1/3})$ and this bound is sharp. As a tool, localization of the quaternion algebras and the representation of p-adic maximal orders as vertices of Serre–Bruhat–Tits trees turn out to be crucial. In low genera, the results shed a surprising new light on the uniformization of some classical curves like Klein's quartic and other Macbeath–Hurwitz curves.

Keywords Dessins d'enfants · Shimura curves · Uniformization · Fuchsian groups · Congruence subgroups · Serre–Bruhat–Tits trees

Mathematics Subject Classification (2000) Primary 20H10; Secondary 11R52 · 14G35 · 20H10 · 30F10 · 30F35

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0 Introduction

A dessin d'enfant can be described as a finite bipartite graph embedded into a compact orientable 2-manifold and dividing it into simply connected cells. Grothendieck [7] observed that a dessin determines a Riemann surface structure, hence an algebraic curve. Rigidity arguments show that the resulting curve can even be defined over a number field (for a recent proof and more references see [6]). Conversely, a theorem of Belyĭ states that every smooth projective algebraic curve X defined over a number field arises in this way [1]. Unfortunately the correspondence is not unique in this direction. For every such curve or—in another terminology—for every such *Belyĭ surface* one can construct infinitely many different dessins.

The situation is easier if one restricts to *quasiplatonic surfaces* and their *regular dessins*. These curves can be defined in many different ways [18], e.g. as quotients $\Gamma\backslash\mathbf{H}$ of the upper half plane $\mathbf{H}$ by a torsion free normal subgroup Γ of a Fuchsian triangle group Δ. The quotient Δ/Γ acts as a group of biholomorphic automorphisms on the surface and transitively on the edges of the dessin, these edges corresponding to the residue classes of Δ mod Γ. As shown in [3] and [5], quasiplatonic surfaces can have finitely many regular dessins only, and these regular dessins are related to each other by conjugations in triangle groups and inclusion relations between them.

The aim of the present paper is to extend these results to *uniform dessins* on Belyĭ surfaces $X = S\backslash\mathbf{H}$, i.e. with a (torsion free) surface group S contained in some triangle group Δ, but no longer necessarily by normal inclusion.

The main general results are contained in Sects. 1–3. There is a remarkable contrast between the study leading to these results in the case of arithmetic and non-arithmetic Fuchsian triangle groups. *In the non-arithmetic case, a Belyĭ surface can have at most four uniform dessins of the same type not equivalent under automorphisms and renormalization. In the arithmetic case, the number of essentially different dessins on a Belyĭ surface X of genus $g \geq 2$ depends on the number and the type of congruence subgroups of Δ containing S. It is bounded from above by $O(g^{1/3})$, and there are series of examples for which this upper bound is attained.*

In Sects. 4–7 we illustrate geometric and arithmetic aspects of these results in low genera and describe explicit examples of curves with different uniform dessins of the same type. In Sect. 6 congruence considerations in quaternion algebras shed a new light on Takeuchi's commensurability diagrams [16] for arithmetic triangle groups.

1 The main question and the answer in the easy case

To put the problem in a precise form we observe first that a surface group S contained in a triangle group Δ is contained in all triangle groups Δ' containing Δ (and maybe also in some triangle subgroups of Δ) inducing dessins of different types on the surface X. All possibilities of such inclusions are well known by work of Singerman [11], so we concentrate on dessins of the same *type* (p, q, r) coming from triangle groups of this signature, i.e. on the following question.

Let S be a Fuchsian surface group contained in a triangle group $\Delta(p, q, r)$. Under which conditions other triangle groups $\Delta'(p, q, r)$ of the same signature contain S and how many of them?

Any two triangle groups of a given signature are conjugate in $\mathrm{PSL}_2\mathbf{R}$, so we can reformulate the problem in the following way.

1. *Let S be a Fuchsian surface group contained in a triangle group $\Delta = \Delta(p, q, r)$. Which and how many different conjugate groups $\alpha^{-1}\Delta\alpha$, $\alpha \in \mathrm{PSL}_2\mathbf{R}$, contain S as well?*

Questions concerning Galois actions on families of dessins often lead to the determination of families of subgroups Γ in a given triangle group Δ having Galois–conjugate quotient curves $X = \Gamma\backslash\mathbf{H}$ with Belyĭ function $\beta : X \to \Delta\backslash\mathbf{H} \cong \hat{\mathbf{C}}$. To determine the *moduli field* of this Belyĭ surface X, i.e. the fixed field of all Galois automorphisms $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$ with the property $X \cong X^\sigma$ one has to determine those $\Gamma < \Delta$ conjugate in $\mathrm{PSL}_2\mathbf{R}$. Therefore the following version of the main problem is interesting as well.

2. *Let Δ be a Fuchsian triangle group and let Γ be a finite index subgroup. For which and for how many $\alpha \in \mathrm{PSL}_2\mathbf{R}$ do we have $\alpha\Gamma\alpha^{-1} < \Delta$?*

Under this condition, conjugation by $\alpha \in \Delta$ induces isomorphisms of both the curve and its dessin, so it is reasonable to count here only residue classes $\alpha \in \mathrm{PSL}_2\mathbf{R}/\Delta$. However, for the first version of the problem it is more natural to count residue classes $\alpha \in N(\Delta)\backslash\mathrm{PSL}_2\mathbf{R}$ where N denotes the normalizer in $\mathrm{PSL}_2\mathbf{R}$.

Definition 1 Let Δ be a Fuchsian group with finite covolume and $\Gamma < \Delta$ a subgroup of finite index. We will denote by $d(\Delta, \Gamma)$ the number of all residue classes $\alpha \in N(\Delta)\backslash\mathrm{PSL}_2\mathbf{R}$ with the property $\Gamma < \alpha^{-1}\Delta\alpha$ (i.e. the number of all groups conjugate to Δ and containing Γ), and by $b(\Delta, m)$ the maximum among all $d(\Delta, \Gamma)$ with index $(\Delta : \Gamma) \leq m$.

If Γ is a surface group and $\Delta = \Delta(p, q, r)$ a triangle group, then $d(\Delta, \Gamma)$ is the number of different uniform dessins of type (p, q, r) on $\Gamma\backslash\mathbf{H}$. The meaning of $b(\Delta, m)$ for the genus is given by the Riemann–Hurwitz formula.

Lemma 1 *Belyĭ surfaces X of genus g can have — up to renormalization — at most $b(\Delta, m)$ uniform dessins of type (p, q, r), where Δ is the triangle group of signature (p, q, r) and*

$$m = \frac{2g - 2}{1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}}.$$

Note that conjugation of Δ by an element $\alpha \in N(\Delta)$ *renormalizes* the dessin on $X = \Gamma\backslash\mathbf{H}$, i.e. permutes the critical values of the Belyĭ function. In any case, α belongs by definition to the *commensurator group* of Γ (and of Δ). Therefore the answer is easy if Δ and hence all subgroups are non-arithmetic Fuchsian groups because then by a theorem of Margulis [8] the commensurator $\overline{\Delta}$ is a finite extension of Δ. In this case, it is well known that $\overline{\Delta}$ is itself a triangle group, and consulting Takeuchi's list of arithmetic triangle groups [16] and Singerman's list of inclusion relations [11] it is easy to see that the index $(\overline{\Delta} : \Delta)$ is at most 6. So we have the first part of

Theorem 1 *Surface groups contained in a non-arithmetic Fuchsian triangle group Δ belong to isomorphic surfaces if and only if they are conjugate in a maximal Fuchsian triangle group $\overline{\Delta}$ extending Δ. They fall in at most 6 different conjugacy classes under conjugation by Δ. If S is such a surface group then $d(\Delta, S) \leq 4$.*

Proof The second part of the theorem follows from the fact that non-normal inclusions $\overline{\Delta} > N(\Delta)$ of non-arithmetic triangle groups occur only with index 3 for $\Delta(2, 3, 2n) > \Delta(2, n, 2n)$ or 4 for $\Delta(2, 3, 3n) > \Delta(3, n, 3n)$, or in the composite cases $\Delta(2, 3, 4n) > \Delta(2, 2n, 4n) \triangleright \Delta(n, 4n, 4n)$, $\Delta(2, 4, 2n) \triangleright \Delta(2, 2n, 2n) \triangleright \Delta(n, 2n, 2n)$.

2 Arithmetic surface groups, localization

Now we concentrate on the remaining case that S and Δ are arithmetic Fuchsian groups, i.e. commensurable to a norm 1 group $\mathcal{M}^1$ of a maximal order $\mathcal{M}$ in a quaternion algebra A defined over a totally real number field k and having precisely one embedding into the matrix algebra $M_2(\mathbf{R})$. The situation here is quite different, as indicated already by the analogous question for normal subgroups in Theorem 3 of [5]. By [16] we know which triangle groups can be identified with the norm 1 group of a maximal order, and most of the arguments will be applied to these cases. However, in general Δ is only commensurable to the norm 1 group of a maximal order in a quaternion algebra, so we will consider in Sect. 6 how far the result is changed by passing to a commensurable group.

Since we have to work in the quaternion algebra A it is often necessary to replace all Fuchsian groups Γ above with their preimages $\hat{\Gamma}$ in $\mathrm{SL}_2\mathbf{R}$. However, if it is clear from the context where the groups are situated, we will often omit the hat to simplify the notation.

We consider now the norm 1 group $\Phi := \mathcal{M}^1$ (which in most cases is a triangle group itself [16]) and restrict our attention to common finite index subgroups S of Φ and $\beta^{-1}\Phi\beta$ and the possible conjugators β in this configuration. Clearly, conjugation by such a β induces an automorphism of the quaternion algebra, therefore the Skolem–Noether theorem ([17], Ch. I, Thm 2.1) allows to replace β with a more convenient element $\alpha \in A$. By multiplication with a denominator in the integers of k we can even suppose α to be in the maximal order $\mathcal{M}$.

Theorem 2 *Let Φ be the norm 1 group of a maximal order $\mathcal{M}$ as above, and suppose $\beta \in \mathrm{SL}_2\mathbf{R}$ such that $\Phi \cap \beta^{-1}\Phi\beta$ have finite index in Φ and $\beta^{-1}\Phi\beta$. Then β can be replaced with a scalar multiple $\alpha \in \mathrm{GL}_2^+\mathbf{R} \cap \mathcal{M} \subset A$.*

Under these conditions $\Phi \cap \alpha^{-1}\Phi\alpha$ is the norm 1 group of an *Eichler order* (i.e. the intersection of two maximal orders in A [17, p. 20]) $\mathcal{M} \cap \alpha^{-1}\mathcal{M}\alpha$. The index of $\Phi \cap \alpha^{-1}\Phi\alpha$ in Φ gives a lower bound for $(\Phi : S)$ where S denotes a surface group contained in both Φ and $\alpha^{-1}\Phi\alpha$. The program is therefore

– to understand how $d(\Phi, S)$ depends on S and
– to determine $(\Phi : \Phi \cap \alpha^{-1}\Phi\alpha) =: s$ as a function of α.

For arithmetic triangle groups one has the additional advantage that all quaternion algebras in question have class number 1 ([16, Prop. 3]), therefore all Eichler orders are intersections of conjugate maximal orders ([17, Ch. I, Cor. 4.11]). So counting multiple dessins on $S\backslash\mathbf{H}$ amounts to count maximal orders containing $\hat{S}$.
Maximal orders are easier to classify locally, i.e. over local fields, and the class number 1 property makes it easy to apply the strong approximation theorem passing to local maximal orders because there are bijections between

– prime ideals in the ring of integers $\mathcal{O}$ of the center k of the quaternion algebra A
– inequivalent primes π in $\mathcal{O}$ generating these prime ideals (without loss of generality we will suppose $\pi > 0$)
– inequivalent discrete valuations v of A
– inequivalent completions $A_v = A_\pi$, $\mathcal{M}_v = \mathcal{M}_\pi$ of the quaternion algebra and a maximal order with respect to v
– two-sided prime ideals in $\mathcal{M}$, all of the form $\pi\mathcal{M}$.

Recall that A_v is a skew field if and only if π ramifies in A, i.e. if it belongs to the finite number of discriminant divisors. In this case, $\mathcal{M}_v$ is the unique maximal order of A_v ([17, Ch. II, Lemme 1.5]), therefore there are no Eichler orders at all. In all other (unramified) cases we get matrix algebras $A_v \cong M_2(k_v)$, $\mathcal{M}_v \cong M_2(\mathcal{O}_v)$ where $\mathcal{O}_v$ denotes the ring of integers in the local field k_v, i.e. the completion of $\mathcal{O}$ in k_v. This ring has the unique prime ideal $\mathcal{P} = \pi\mathcal{O}_v$, and all Eichler orders are conjugate to a ring of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad a, b, d \in \mathcal{O}_v, \ c \in \mathcal{P}^n$$

for some positive integer n ($\mathcal{P}^n$ is the *level* of the Eichler order). This local Eichler order is in fact an intersection $\mathcal{M}_v \cap \alpha^{-1}\mathcal{M}_v\alpha$ of two maximal orders conjugate by some $\alpha \in \mathcal{M}_v^* \begin{pmatrix} \pi^n & 0 \\ 0 & 1 \end{pmatrix} \subset M_2(\mathcal{O}_v)$.

The strong approximation theorem allows now to trace back all considerations about conjugations or Eichler orders to simultaneous localizations. We need in particular the following two consequences.

Fact 1 *For all inequivalent discrete valuations v of the quaternion algebra A let $\mathcal{M}_v$ be a maximal order in A_v. Suppose that $\mathcal{M}_v = M_2(\mathcal{O}_v)$ for almost all v. Then there is a maximal order $\mathcal{M}$ in A such that all $\mathcal{M}_v$ are the localizations (= completions in the v–adic topology) of $\mathcal{M}$.*

Fact 2 *For all inequivalent discrete valuations v of the quaternion algebra A let S_v be a group commensurable to the norm* 1 *group $\mathcal{M}_v^1$. Suppose $S_v = \mathcal{M}_v^1$ for almost all v. Then there is a multiplicative group $S \subset A^*$ commensurable to the global norm* 1 *group $\mathcal{M}^1 \subset A^*$ whose localizations are the groups S_v.*

Definition 2 Let the Fuchsian group Φ (more precisely, its $\mathrm{SL}_2(\mathbf{R})$–preimage) be commensurable to the norm 1 group $\mathcal{M}^1$ of a maximal order $\mathcal{M}$ in a quaternion algebra A of class number 1 and let π be a prime in the totally real field k, the center of A. For a subgroup $S < \Phi$ of finite index let $S_\pi := S_v$ and $\Phi_\pi := \Phi_v$ be their closure (v–completion) in A_v where v is the discrete valuation corresponding to π.

We will denote by $d_\pi(\Phi, S)$ the number of all local conjugates $\alpha^{-1}\Phi_\pi\alpha$ containing S_π, and by $b_\pi(\Phi, m_\pi)$ the maximum of all those $d_\pi(\Phi, S)$ for which the v–completion S_π of S has index $(\Phi_\pi : S_\pi) \leq m_\pi$.

In the case of $\Phi = \mathcal{M}^1$ one may define $d_\pi(\Phi, S)$ also as the number of maximal orders $\alpha^{-1}\mathcal{M}_v\alpha$ containing S_π.

For almost all π the v–completion is $S_\pi = \Phi_\pi$, hence $d_\pi(\Phi, S) = 1$. Therefore the first product in Theorem 3 below has only finitely many factors $\neq 1$. We will see in Remark 1 (next section) that the same property holds for the product $b(\Phi, m)$ because for every $M > 1$ there is only a finite number of possible π with $1 < m_\pi < M$, and for $m_\pi = 1$ we have clearly $b_\pi(\Phi, 1) = 1$. Up to this finiteness property Theorem 3 follows directly with simultaneous localization and the consequences of the strong approximation theorem stated above.

Theorem 3 *For a finite index subgroup S of the arithmetically defined Fuchsian group Φ and under these notations we have*

$$d(\Phi, S) = \prod d_\pi(\Phi, S) \quad \text{and} \quad b(\Phi, m) = \max_{\prod m_\pi \leq m} \prod b_\pi(\Phi, m_\pi)$$

where all products run over the inequivalent primes of k not dividing the discriminant of A, and the maximum runs over all infinite sequences $(m_\pi)_\pi$ of positive integers indexed by these inequivalent primes.

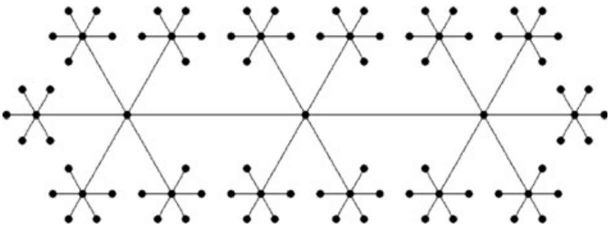


Fig. 1 Part of the tree of local maximal orders for $q = 5$

3 The local situation

Now we have to determine these factors in the localized quaternion algebras of type $A_v \cong M_2(k_v)$. We omit the ramified primes because for them we have $d_\pi(\Phi, S) = 1$, hence $b_\pi(\Phi, m_\pi) = 1$ for all $m_\pi > 1$. We begin with a study of the Eichler orders of level $\mathcal{P}$ (omitting the hat again).

For this study it will be helpful to consider the *tree of maximal orders* as described in [17, pp. 40–41]. Maximal orders of a split local quaternion algebra A_v correspond to the vertices of a tree. Two vertices are joined by an edge if and only if the corresponding maximal orders are conjugate under an element whose norm is in $\mathcal{O}_v^*\pi$, and the tree is regular with valency $q+1$ in every vertex (see Fig. 1) where the norm $q = N(\pi)$ denotes the number of elements of the residue class field $\mathbf{F}_q = \mathcal{O}_v/\mathcal{P}$.

We can identify Eichler orders of level $\mathcal{P}$ with edges and Eichler orders of level $\mathcal{P}^n$ with paths of length n in the tree joining $\mathcal{M}_v$ with other vertices (maximal orders $\alpha^{-1}\mathcal{M}_v\alpha,\ \alpha$ of norm π^n). As can be seen in

$$\mathcal{M}_v > \mathcal{M}_v \cap \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathcal{M}_v \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} > \cdots > \mathcal{M}_v \cap \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}^{-n} \mathcal{M}_v \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}^{n},$$

an Eichler order is contained in all the maximal orders corresponding to vertices lying in the path. By the same reason, we have

Lemma 2 *Let S be a finite index subgroup of the norm 1 group Φ of the maximal order $\mathcal{M}$. Then all vertices corresponding to local maximal orders Φ_π-conjugate to $\mathcal{M}_v$ containing S_π form the vertices of a finite subtree, and $d_\pi(\Phi, S)$ counts the vertices of this subtree.*

Definition 3 Let S_π be a finite index subgroup of Φ_π. We denote by $\mathcal{T}(S_\pi)$ the subtree of maximal orders of the local quaternion algebra A_π whose vertices correspond to the maximal orders containing S_π.

We will see in the following how these subtrees can look like. We begin with the simplest cases. In the rest of this section, we will consider only the local situation, so we omit the index π as long as the groups are concerned.

Lemma 3 *Let Φ be the norm 1 group of the local maximal order $\mathcal{M}_v = M_2(\mathcal{O}_v)$, $\mathcal{O}_v$ the ring of integers in the local field k_v with maximal ideal $\mathcal{P}$ and residue class field $\mathcal{O}_v/\mathcal{P} = \mathbf{F}_q$. Now we consider Φ and its subgroups as subgroups of* $\mathrm{PSL}_2(\mathcal{O}_v)$*, i.e. modulo* $\pm\mathrm{Id}$. *Then*

1. *the norm 1 group $\Phi_0 = \Phi_0(\mathcal{P})$ of an Eichler order of level $\mathcal{P}$ has index $q+1$ in Φ, and for these groups, $d_\pi(\Phi, \Phi_0) = 2$.*

2. *The norm* 1 *group* $\Phi_0^0 = \Phi_0^0(\mathcal{P})$ *of the intersection of two Eichler orders of level* $\mathcal{P}$ *has index* $q(q+1)$ *in* Φ, *and for these groups,*

$$d_\pi(\Phi, \Phi_0^0) = 3 \quad \text{if} \quad q > 3,$$
$$d_\pi(\Phi, \Phi_0^0) = 4 \quad \text{if} \quad q = 2 \quad \text{and} \quad d_\pi(\Phi, \Phi_0^0) = 5 \quad \text{if} \quad q = 3.$$

3. *The norm* 1 *group* $\Phi(\mathcal{P})$ *in the intersection of more than two Eichler orders of level* $\mathcal{P}$ *is the principal congruence subgroup* $\bmod \mathcal{P}$ *of* Φ, *a normal subgroup of* Φ *of index* $\frac{1}{2}q(q^2-1)$ (*omit the denominator* 2 *if* q *is a* 2*–power*). *It is the intersection of all such Eichler orders of level* $\mathcal{P}$ *and satisfies* $d_\pi(\Phi, \Phi(\mathcal{P})) = q + 2$.

Proof The proof is easy if one considers the canonical operation of Φ on the projective line $\mathbf{P}^1(\mathbf{F}_q)$ given by reduction $\bmod \mathcal{P}$. In this frame, the groups Φ_0 are the subgroups fixing one point, Φ_0^0 are those fixing two points, and if more than two points are fixed, all points of the projective line are fixed, hence the last case gives already the principal congruence subgroup. Recall that d_π is always 1+ the number of Eichler orders involved since we have to count $\mathcal{M}_v$ as well. The cases $q = 2$ and 3 play a special role because for them $\Phi_0^0(\pi) = \Phi(\pi)$: recall that we see them as projective groups, and since the determinants are 1, in the case of small q all matrices in $\Phi_0^0(\pi)$ are congruent $\bmod \pi$ to $\pm$ the unit matrix.

For the calculation of the indices one may consult [17, p. 109] or mimic a proof from any book about modular forms. Alternatively one may consider the groups involved as the stabilizers of one point, two points or the whole projective line, and then the index is given by the number of elements in the orbit of the fixpoints. □

Lemma 4 *For integers* $n > 1$ *there are* $q^{n-1}(q+1)$ *different local Eichler orders* $\mathcal{M}_v \cap \alpha^{-1}\mathcal{M}_v\alpha$ *of level* $\mathcal{P}^n$. *Their norm* 1 *groups* $\Phi_0(\mathcal{P}^n)$ *have index* $q^{n-1}(q+1)$ *in* Φ. *They satisfy*

$$d_\pi(\Phi, \Phi_0(\mathcal{P}^n)) = n + 1 \quad \text{for} \quad q > 3,$$
$$d_\pi(\Phi, \Phi_0(\mathcal{P}^n)) = 3n - 1 \quad \text{for} \quad q = 3,$$
$$d_\pi(\Phi, \Phi_0(\mathcal{P}^n)) = 2n \quad \text{for} \quad q = 2,\ n = 2 \text{ or } 3 \quad \text{and}$$
$$d_\pi(\Phi, \Phi_0(\mathcal{P}^n)) \geq 4n - 6 \quad \text{for} \quad q = 2,\ n \geq 4.$$

The intersection of all these norm 1 *groups is the principal congruence subgroup* $\Phi(\mathcal{P}^n)$ *and satisfies* $d_\pi(\Phi, \Phi(\mathcal{P}^n)) = \frac{(q+1)(q^n-1)}{q-1} + 1$.

Proof To prove that there are precisely $q^{n-1}(q+1)$ such Eichler orders of level $\mathcal{P}^n$ with norm 1 group $\Phi_0(\mathcal{P}^n)$ one may just count paths of length n in the tree of maximal orders, with one end fixed in the vertex $\mathcal{M}_v$. Therefore, following the unique path in the tree of maximal orders we find a corresponding unique chain of Eichler orders proving the claim about their numbers. For $q > 3$ the number $q + 1$ of vertices on this path gives also the number of maximal orders containing $\Phi_0(\mathcal{P}^n)$ since otherwise $\Phi_0(\mathcal{P}^n)$ would be contained in more than one Eichler order $\mathcal{M}_v \cap \alpha^{-1}\mathcal{M}_v\alpha$ of level $\mathcal{P}^m$ for some $m \leq n$. To see that this is impossible, one can generalize the argument sketched in the proof of Lemma 3 defining a kind of "projective line" $\mathbf{P}^1_m$ over the residue class ring $\mathcal{O}/\pi^m \cong \mathcal{O}_v/\mathcal{P}^m$ as set of pairs of residue classes, not both in $\pi\mathcal{O}/\pi^m$, modulo the unit group of this residue class ring. The norm 1 group Φ acts in a natural way on this $\mathbf{P}^1_m$, and its subgroup $\Phi_0(\mathcal{P}^n)$ has precisely one fixed point on it if $q > 3$, in contrast to all groups of type $\Phi_0^0(\mathcal{P}^m)$ having at least two fixed points on $\mathbf{P}^1_m$. So one has in fact $d_\pi(\Phi, \Phi_0(\mathcal{P}^n)) = n + 1$. All but the final among these Eichler orders belong to lower levels, so by induction on n one gets the result about $d_\pi(\Phi, \Phi(\mathcal{P}^n))$ for $q > 3$. For the index formula one may use the same argument of the

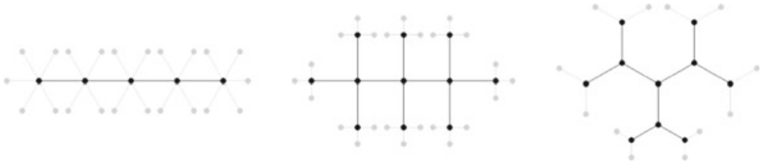


Fig. 2 Subtree for $\Phi_0(\pi^4)$, in the cases $q = 5, 3$ and 2

previous Lemma, this time considering the action on $\mathbf{P}^1_n$, or use [17, p. 55]. For $q = 2$ and 3 it is no longer true that $\Phi_0(\mathcal{P}^n)$ has precisely one fixed point on $\mathbf{P}^1_m$: an exercise in congruences shows that

$$\Phi_0(\mathcal{P}^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Phi \subset M_2(\mathcal{O}_v) \mid c \equiv 0 \bmod \pi^n \right\}$$

fixes not only $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbf{P}^1_m$ but also all $\begin{bmatrix} 1 \\ x \end{bmatrix}$, $x \equiv 0 \bmod \pi^{m-1}$, and for $q = 2,\ m \geq 4$ moreover $\begin{bmatrix} 1 \\ x \end{bmatrix}$ for all $x \equiv 0 \bmod \pi^{m-2}$. As indicated in Fig. 2 the subtree of maximal orders containing $\Phi_0(\mathcal{P}^n)$ is therefore larger than the simple path joining two extremal vertices as in the case $q > 3$. However, the index formula and the result about $d_\pi(\Phi, \Phi(\mathcal{P}^n))$ remain true also in these cases. (A more detailed case by case analysis shows that in the case of cocompact arithmetic triangle groups the last assertion is even true with "=" instead of "≥".) □

As an illustration for the result concerning the principal congruence subgroups, we give here the picture of the subtree for $\Phi(\pi^2)$ in the case $q = 7$.

Remark 1 For fixed m only a finitely many primes π in k lead to a residue class field with $q < m$, so the products in Theorem 3 are well defined.

Lemma 5 *Let G be a finite index subgroup of the norm* 1 *group Φ in a local maximal order $\mathcal{M}_v$ of the quaternion algebra A_v. Then the subtree $\mathcal{T}(G)$ is determined by two integers $n, k \geq 0$ in the following way. There is a simple path C of length k in $\mathcal{T}(G)$ such that the vertices of $\mathcal{T}(G)$ consist precisely of those vertices in the tree of local maximal orders having distance $\leq n$ from C, the "spine" of $\mathcal{T}(G)$.*

Figure 1 gives an illustration of such a subtree for $n = k = 2$ and $q = 5$, Fig. 2 gives such subtrees for $(n, k) = (0, 4),\ (1, 2)$ and $(2, 0)$ in the cases $q = 5, q = 3$ and $q = 2$, and Fig. 3 gives another example of $(n, k) = (2, 0)$, this time for $q = 7$.

Proof 1. Let n be the maximal integer such that there is a vertex v whose full distance n neighbourhood belongs to $\mathcal{T}(G)$, in other words with the property that all other vertices of distance $\leq n$ in the tree of all local maximal orders belong to $\mathcal{T}(G)$. If $\mathcal{T}(G)$ has no other vertex outside this neighbourhood of radius n around v, the claim is true with $k = 0$.

2. If $\mathcal{T}(G)$ contains more vertices than those of distance $\leq n$ from $v = v_0$, it contains a vertex v_{n+1} of distance $n + 1$ since $\mathcal{T}(G)$ is a subtree by Lemma 2. The next vertex v_1 on the simple path from v_0 to v_{n+1} has then also the property that all vertices with distance $\leq n$ from v_1 belong to $\mathcal{T}(G)$. In fact, suppose that v_0 corresponds to the standard maximal order $M_2(\mathcal{O}_v)$, then we can suppose via conjugation in Φ that G is contained

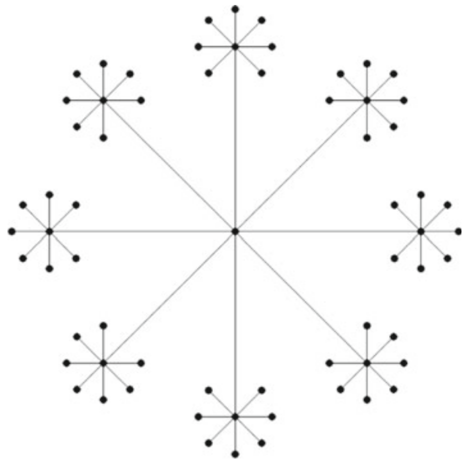


Fig. 3 Subtree for $\Phi(\pi^2)$ in the local algebra A_π for $q = 7$

in $\Phi(\pi^n) \cap \Phi_0(\pi^{n+1})$. This group is contained in q^n Eichler orders $\mathcal{M}_v \cap \alpha^{-1}\mathcal{M}_v\alpha$ of level $\mathcal{P}^{n+1}$: as in the proof of Lemma 4 consider its action on the generalized projective line $\mathbf{P}^1_{n+1}$; if $a \equiv d \equiv 1 \bmod \pi^n$, $b \equiv 0 \bmod \pi^n$ and $c \equiv 0 \bmod \pi^{n+1}$ with $ad - bc = 0$, then another easy exercise in congruences shows that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ not only fixes the point $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbf{P}^1_{n+1}$ but all q^n points $\begin{bmatrix} 1 \\ x \end{bmatrix}$ with $x \equiv 0 \bmod \pi$. This means in particular that the subtree $\mathcal{T}(\Phi(\pi^n) \cap \Phi_0(\pi^{n+1}))$ has q^n more vertices than $\mathcal{T}(\Phi(\pi^n))$.

3. Apparently, v_0 and v_1 are both vertices of the spine C which may be constructed by an obvious continuation of this idea. Suppose we have already a simple path of length $m > 0$ whose vertices w_j, $j = 0, \ldots, m$, belong to $\mathcal{T}(G)$ together with all their neighbours of distance $\leq n$. If $\mathcal{T}(G)$ has no other vertex, we have already the desired subtree with $m = k$.
4. If not, $\mathcal{T}(G)$ contains a vertex v' of distance $> n$ from all w_j. Then choose that w_j with minimal distance from v'—it is in fact unique since we have a tree—join it to v' by a path inside $\mathcal{T}(G)$ and apply part 2 of the proof again to see that the neighbour of w_j on this path is again a vertex of C.
5. Now there could be two possibilities. First, w_j is a boundary point of the path we already found, say w_0 or w_m. In this case, the new neighbour (w_{-1} or w_{m+1}, say) continues the path C.
6. Second, the new vertex w of C could be a third neighbour of w_j together with w_{j-1} and w_{j+1}. This case is impossible by a similar reason as given in the proof of Lemma 3, part 3: Let w_j denote the standard maximal order $M_2(\mathcal{O}_v)$. Since C contains three different neighbours of w_j and all their neighbours of distance $\leq n$, we can suppose that G is contained in $\Phi(\pi^n) \cap \Phi_0^0(\pi^{n+1})$ and fixes—acting on $\mathbf{P}^1_{n+1}$—moreover also some point $\begin{bmatrix} y \\ x \end{bmatrix}$ where both x and y are note divisible by π. But in that case we have $G \leq \Phi(\pi^{n+1})$ meaning that all vertices of distance $\leq n + 1$ from w_j belong to $\mathcal{T}(G)$ in contradiction to our choice of n.

As a side result of this proof we note the number of vertices in $\mathcal{T}(G)$.

Lemma 6 *Under the same hypotheses as in Lemma 5 the number of vertices in $\mathcal{T}(G)$ is* $d_\pi(\Phi, G) = 1 + \frac{(q+1)(q^n-1)}{q-1} + kq^n$.

Remark 2 For $q > 3$ all choices of the parameters n and k are possible: take $G := \Phi(\pi^n) \cap \Phi_0(\pi^{n+k})$ in the Lemmas 5 and 6. The cases $q = 2$ and 3 behave differently as already seen in Lemma 4 and its proof: for $k \geq 2$ we have $\Phi_0(\pi^2) \cong \Phi(\pi)$ hence $n \geq 1$; and for $q = 2$, $k \geq 4$ we have even $n \geq 2$. In fact, one may prove that in the latter case $\Phi_0(\pi^4)$ is conjugate in the local algebra to $\Phi(\pi^2)$. Take Fig. 2 as an illustration of the subtrees $\mathcal{T}(G)$ for the groups $G = \Phi_0(\pi^4)$ and $\Phi_0^0(\pi^2)$ as well.

For the next lemma we note first—generalizing the statement of Lemma 3, part 3—that the index of the principal congruence subgroup $\Phi(\mathcal{P}^n)$ itself in Φ is

$$i(\pi, n) := \frac{1}{2} \cdot q^{3n-2}\left(1 - \frac{1}{q^2}\right) \qquad \left(\text{without the factor } \frac{1}{2} \text{ if } q \text{ is a 2–power}\right).$$

To perform induction over the levels, observe that for all $n > 0$ the quotient $\Phi(\mathcal{P}^n)/\Phi(\mathcal{P}^{n+1})$ is abelian, more precisely isomorphic to the additive group of the vector space $\mathbf{F}_q^3$. By the same arguments we obtain for $n, k > 0$ the index formula

$$(\Phi : (\Phi(\pi^n) \cap \Phi_0(\pi^{n+k})) = i(\pi, n) \cdot q^k.$$

Lemma 7 *If $q > 3$,*

$$b_\pi(\Phi, i(\pi, n)) = \frac{(q+1)(q^n-1)}{q-1} + 1$$

for all positive integers n.

Proof We have to show that besides the principal congruence subgroups of level $\mathcal{P}^n$ all other subgroups $S \subset \Phi$ of index $\leq i(\pi, n)$ are contained in a smaller number of maximal orders. We can suppose that S is already a norm 1 subgroup G of an intersection of Eichler orders as discussed in Lemmas 5 and 6. Since $i(\pi, n)$ grows with q^{3n} and $d_\pi(\Phi, G)$ for subgroups of index $i(\pi, n')q^k \leq i(\pi, n)$ grows at most like $(1 + k) \cdot q^{n'}$, the maximal d_π is certainly obtained for $n = n'$, $k = 0$ i.e. for the principal congruence subgroups.

4 Global consequences

It remains to insert the results of the previous section in Theorem 3 and to illustrate these by examples. We begin with an obvious necessary condition for the existence of at least two different uniform dessins of the same type on a Riemann surface of genus > 1, crucial for the construction of low genus examples.

Theorem 4 *Let S be an arithmetic Fuchsian surface group contained in the triangle group Δ, and suppose Δ to be the norm 1 group $\mathcal{M}^1$ in a maximal order $\mathcal{M}$ of a quaternion algebra A defined over the totally real field k with ring of integers $\mathcal{O}$. The group S is contained in more than one group conjugate to Δ in $\mathrm{PSL}_2\mathbf{R}$ if and only if S is contained in a group conjugate in Δ to*

$$\Delta_0(\pi) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta \subset M_2(\mathcal{O}) \mid c \equiv 0 \bmod \pi \right\}$$

where π is a prime of k not dividing the discriminant of A.

Given S and Δ as above one can use Theorem 3 and Lemma 6 to compute the number $d(\Delta, S)$ of different dessins on $S\backslash\mathbf{H}$.

Now we will concentrate on a series of striking examples. Take Δ of signature (2, 3, 7). According to [16] this is the norm 1 group of a maximal order $\mathcal{M}$ in a quaternion algebra A over the cubic field $k = \mathbf{Q}(\cos\frac{2\pi}{7})$. All primes π of k are unramified in A. By a recent result of Džambić [2] all Macbeath–Hurwitz groups—giving the most famous examples of Hurwitz surfaces—are principal congruence subgroups in Δ. The first cases are

- Klein's quartic. Its surface group is $\Delta(\pi)$ for a prime π dividing 7, ramified of order 3 and of residue degree 1 in the extension $\mathbf{Q}(\cos\frac{2\pi}{7})/\mathbf{Q}$. With $q = 7$ we see that Klein's quartic has 8 non-conjugate uniform dessins of type (2, 3, 7) plus the usual regular one.
- Macbeath's curve of genus 7 with automorphism group $\mathrm{PSL}_2(\mathbf{F}_8)$ has the surface group $\Delta(2)$ for the prime $\pi = 2$, inert and of residue degree 3 in the extension $\mathbf{Q}(\cos\frac{2\pi}{7})/\mathbf{Q}$. With $q = 8$ one has 9 uniform dessins plus a regular one on the curve.
- Three non-isomorphic curves in genus 14 with automorphism group $\mathrm{PSL}_2(\mathbf{F}_{13})$ whose surface groups are the principal congruence subgroups $\Delta(\pi_j)$, $j = 1, 2, 3$ for the (completely decomposed) primes π_j dividing 13. Their residue degree is 1, hence one has $q + 1 = 14$ uniform dessins of type (2, 3, 7) on each curve plus a regular one.

All dessins mentioned here are clearly not renormalizations of each other since the signature consists of three different entries. On the other hand, in all these cases we have one regular dessin and $q + 1$ uniform non-regular ones which form an orbit under the automorphism group of the curve: the $q + 1$ norm 1 groups of type $\Delta_0(\pi)$ are conjugate under the action of Δ or—in other words—the $q + 1$ Eichler orders of level $\mathcal{P}$ form a Δ-invariant set, so these dessins are equivalent under automorphisms of the curve.

Up to conjugation, we have therefore only the rather modest number of two essentially different dessins of the same type. Similarly, even if $\Delta_0(\pi)$ in Theorem 4 was torsion free, the two different dessins on the curve $\Delta_0(\pi)\backslash\mathbf{H}$ are equivalent under the action of the *Fricke involution*, conjugate in $M_2(\mathcal{O}_v)$ to $\rho = \begin{pmatrix} 0 & \pi \\ -1 & 0 \end{pmatrix}$, and two among the three dessins on a curve $\Delta_0^0(\pi)\backslash\mathbf{H}$ are equivalent under an involution as well, conjugate in $M_2(\mathcal{O}_v)$ to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

However replacing these congruence groups with subgroups of small index we can remove automorphisms such that most of the uniform dessins found here become inequivalent under automorphisms, see Remark 3 below.

Next we consider the growth of the maximal number of uniform dessins on surfaces $S\backslash\mathbf{H}$ depending on the index $(\Delta : S)$ in a given triangle group.

Theorem 5 *Let the Fuchsian group Φ be the norm* 1 *group of a quaternion algebra. Then*

$$b(\Phi, m) = O(\sqrt[3]{m})$$

and this upper bound is optimal in the following sense. There are sequences of surface groups $S_n < \Phi$ with indices $(\Phi : S_n) \to \infty$ such that for the numbers $d(\Phi, S_n)$ of all residue classes $\alpha \in \mathrm{PSL}_2\mathbf{R}/N(\Phi)$ with the property $S_n \subset \alpha^{-1}\Phi\alpha$ we have

$$\lim_{n\to\infty} \frac{d(\Phi, S_n)}{\sqrt[3]{2(\Phi : S_n)}} = 1.$$

Proof The proof follows from the last part of Theorem 3 if we use the fact that the index $m = (\Phi : S)$ grows at least like the product of all indices m_π of the localizations, and for these we can apply Lemma 6 together with the formula for the indices of the principal congruence subgroups. For the sequence S_n one may take any sequence of principal congruence subgroups $\Phi(\mathcal{P})$ with prime ideals $\mathcal{P} = \pi\mathcal{O}_v$ such that $\mathcal{O}/\pi \cong \mathbf{F}_q,\ q \to \infty$. Observe that only finitely many among the S_n can have torsion.

Remark 3 One may replace this sequence S_n of surface groups with a sequence G_n whose normalizers are by far smaller. We start with the principal congruence subgroups $\Phi(\pi^n)$ where we suppose for simplicity (satisfied in the case of arithmetic triangle groups) that

- the center field k has class number 1,
- π is a prime in k not dividing the discriminant of the algebra A,
- dividing a prime $p \in \mathbf{Z},\ p > 3$, split in k (infinitely many such primes exist by Dirichlet's prime number theorem),
- hence with norm $q = p$,
- and n so large that $\Phi(\pi^n)$ is torsion free of genus $g = g_n$.

The index of $\Phi(\pi^n)$ in the norm 1 group $\Phi = \mathcal{M}^1$ is $\frac{1}{2}(p^2-1)p^{3n-2}$, and the normalizer in $\mathrm{PSL}_2\mathbf{R}$ of any finite index subgroup $G_n < \Phi(\pi^n)$ is contained in the unit group A^*, more precisely in its projective image PA^*, see the Skolem–Noether arguments used in the proof of Theorem 2. By the same reasons we can represent all elements of $N(G_n)$ by elements of $A^* \cap \mathcal{M}$. Dividing out unnecessary factors we can moreover suppose that these elements are not divisible by primes ρ of $\mathcal{O}_k$. If two such elements α, β fall in the same residue class in $PA^*/\Phi(\pi)$, they satisfy $\sigma\alpha \equiv \tau\beta \mod \pi\mathcal{M}$ for some coprime $\sigma, \tau \in \mathcal{O}_k$ not divisible by π (otherwise α or β would be divisible by π). The number of these residue classes is therefore less than p^4, the number of residue classes in $\mathcal{M}/\pi\mathcal{M} \cong M_2(\mathcal{O}_k/\pi\mathcal{O}_k)$.

We note first that the group isomorphism

$$N(G_n)\Phi(\pi)/\Phi(\pi) \cong N(G_n)/(N(G_n) \cap \Phi(\pi))$$

and the above count of residue classes gives

$$(N(G_n) : (\Phi(\pi) \cap N(G_n))) \leq (PA^* : \Phi(\pi)) \ < \ p^4.$$

Second, if we succeed to construct G_n in such a way that $N(G_n) \cap \Phi(\pi) = N(G_n) \cap \Phi(\pi^n)$, there is a bijection between the residue classes of $N(G_n) \mod \pi^n$ and those of $N(G_n) \mod \pi$. If so, we can deduce

$$\begin{aligned}(N(G_n) : G_n) &\leq (N(G_n) : (\Phi(\pi^n) \cap N(G_n))) \cdot (\Phi(\pi^n) : G_n)\\ &\leq (N(G_n) : (\Phi(\pi) \cap N(G_n))) \cdot (\Phi(\pi^n) : G_n) < p^4 \cdot (\Phi(\pi^n) : G_n).\end{aligned}$$

As third step it remains therefore to construct a subgroup G_n of of small index in $\Phi(\pi^n)$ such that $N(G_n) \cap \Phi(\pi) = N(G_n) \cap \Phi(\pi^n)$. We begin with the observation that the group $\Phi(\pi)/\Phi(\pi^n)$ is generated by three elements of order p^{n-1}, namely

$$\begin{pmatrix} 1 & 0 \\ \pi & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pi \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \text{ with } a = 1 + \pi,\ d = 1 - \pi + \pi^2 - \cdots \equiv a^{-1} \mod \pi^n.$$

In fact, $\Phi(\pi)/\Phi(\pi^n)$ is a p–group, every element γ has an order p^k such that $\gamma^{p^{k-1}} \in \Phi(\pi^{n-1})/\Phi(\pi^n)$ is an element of order p. As already mentioned in the last section, this quotient group $\Phi(\pi^{n-1})/\Phi(\pi^n)$ is abelian and isomorphic to the additive group of $\mathbf{F}_p^3$.

Recall that $\Phi(\pi^n)$ is torsion free, generated by $a_j, b_j,\ j = 1, \ldots, g$ with the single generating relation $\prod[a_j, b_j] = 1$, and that these generators correspond to the generators of the homology of the surface $X_1 := \Phi(\pi^n)\backslash\mathbf{H}$, isomorphic to $\mathbf{Z}^{2g}$. Let γ_1 one of the above generators of of $\Phi(\pi)/\Phi(\pi^n)$ of order p^{n-1} and $\delta_1 := \gamma_1^{p^{n-2}}$. Conjugation of $\Phi(\pi^n)$ by the elements of $N(\Phi(\pi^n))$ induces an effective action of the automorphism group of the surface on its homology, and since the subgroups $\langle\gamma_1\rangle > \langle\delta_1\rangle$ have odd order, they act effectively on the quotient $(\mathbf{Z}/2\mathbf{Z})^{2g}$ as well: consider the complex representation and its corresponding modular representation over an algebraic closure of $\mathbf{F}_2$. Therefore there is a generator — a_1, say — sent by δ_1 to some $\delta_1(a_1) \not\equiv a_1 \bmod 2$. Looking at generators and relations of $\Phi(\pi^n)$ there is hence a character

$$\chi_1 : \Phi(\pi^n) \to \{\pm 1\} \quad \text{with} \quad \chi_1(a_1) = -1,\ \chi_1(\delta_1(a_1)) = 1$$

whose kernel is not invariant under δ_1, therefore not invariant under any nontrivial element of the cyclic group $\langle\gamma_1\rangle$. So the intersection $(N(\mathrm{Ker}\ \chi_1) \cap \Phi(\pi^{n-1}))/\Phi(\pi^n)$ can be an at most 2-dimensional $\mathbf{F}_p$–vector space. If it is nontrivial, take an element $\gamma_2 \in N(\mathrm{Ker}\ \chi_1) \cap \Phi(\pi)$ of maximal order $p^k > 1$, define $\delta_2 := \gamma_2^{p^{k-1}} \in \Phi(\pi^{n-1})$ and consider its action on the homology of the surface $X_2 := \mathrm{Ker}\ \chi_1 \backslash \mathbf{H}$. Again it is effective and we can define a character $\chi_2 : \mathrm{Ker}\ \chi_1 \to \{\pm 1\}$ whose kernel is not invariant under conjugation by δ_2 and a fortiori not under conjugation by γ_2 nor δ_1 nor γ_1. Now the intersection $(N(\mathrm{Ker}\ \chi_2) \cap \Phi(\pi^{n-1}))/\Phi(\pi^n)$ is at most onedimensional in $\mathbf{F}_p^3$ and we can finish the construction by taking a $\gamma_3 \in N(\mathrm{Ker}\ \chi_2) \cap \Phi(\pi)$ of maximal order mod $\Phi(\pi^n)$ and defining a $\chi_3 : \mathrm{Ker}\ \chi_2 \to \{\pm 1\}$ as above to be sure that $N(\mathrm{Ker}\ \chi_3) \cap \Phi(\pi) \leq \Phi(\pi^n)$.

Define $G_n := \mathrm{Ker}\ \chi_3$ (or $\mathrm{Ker}\ \chi_2$ or $\mathrm{Ker}\ \chi_1$ if there is no nontrivial γ_3 or γ_2, respectively). This group satisfies our hypothesis in the second step, hence $(N(G_n) : G_n) < 8p^4$. Therefore, the surface $G_n\backslash\mathbf{H}$ has less than $8p^4$ automorphisms and, by Theorem 3 together with Lemma 7 more than

$$\frac{1}{8p^4}\left((p+1)\cdot\frac{p^n-1}{p-1}+1\right)$$

uniform dessins inequivalent under automorphisms.

As in Lemma 1, we can describe the growth result given in Theorem 5 also in terms of the genus, by Remark 3 now in a stronger version:

Corollary 1 *The number of uniform dessins not equivalent under renormalization or automorphisms on a Belyĭ surface grows with the genus g at most as a multiple of $\sqrt[3]{g}$, and this bound is optimal.*

5 A geometrical description

We explore now the examples given in Sect. 4 in a more geometrical way.

Klein's quartic. Klein's quartic is a genus three surface uniformized by a group S generated by certain side-pairings in the regular 14-gon P with angle $2\pi/7$ (see Fig. 4). The (black and white) triangles in Klein's original picture are related to the triangle group $\Delta(2, 3, 7)$ of signature $(2, 3, 7)$ in which S is normally contained with index 168.

The inclusion $S \lhd \Delta(2, 3, 7)$ induces a regular Belyĭ function on S. The corresponding regular dessin $\mathcal{D}$ can be easily depicted in P with the help of the triangle tessellation associated to $\Delta(2, 3, 7)$ (see left picture on Fig. 5).

Rotate now $\mathcal{D}$—or rather its lift to the universal covering $\mathbf{D}$—by an angle $2\pi/14$ around the origin. The graph $\mathcal{D}'$ obtained is compatible with the side-pairing identifications, hence

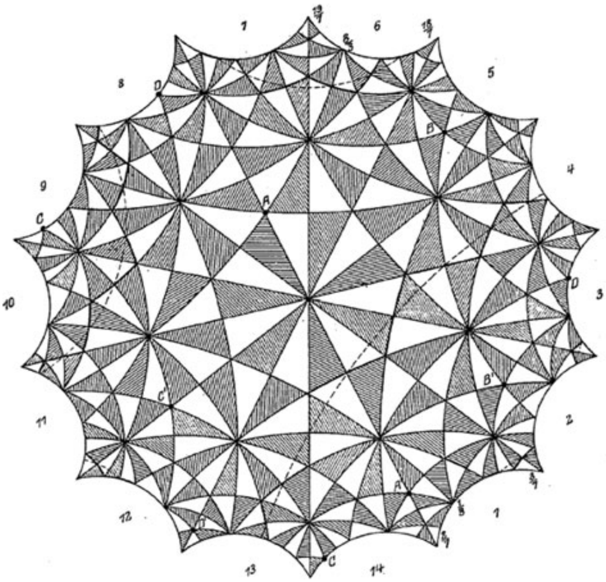


Fig. 4 Klein's surface is obtained by the side pairing $1 \leftrightarrow 6, 3 \leftrightarrow 8, 5 \leftrightarrow 10, 7 \leftrightarrow 12, 9 \leftrightarrow 14, 11 \leftrightarrow 2, 13 \leftrightarrow 4$

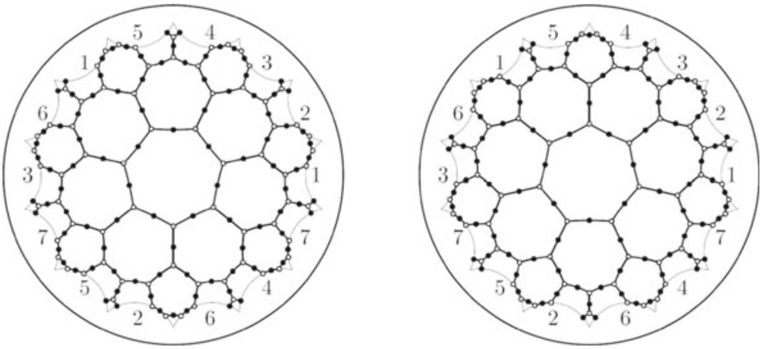
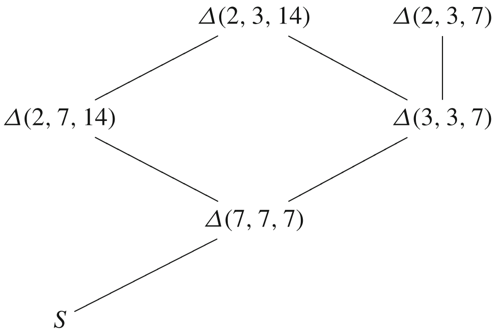


Fig. 5 Klein's regular (2, 3, 7) dessin $\mathcal{D}$ and a uniform one $\mathcal{D}'$

it is a well defined dessin on the surface. It is rather obvious that $\mathcal{D}'$ decomposes the surface into 24 heptagons in the same way as $\mathcal{D}$ does. In other words $\mathcal{D}'$ is also a uniform (2, 3, 7) dessin on $S\backslash\mathbf{H}$ (see right picture on Fig. 5). Note that the rotation that transforms $\mathcal{D}$ into $\mathcal{D}'$ does not correspond to any automorphism of the surface, and in fact both dessins are not isomorphic since $\mathcal{D}'$ is not regular.

The existence of a new uniform dessin of type (2, 3, 7) is clear if one studies all triangle groups in which S is contained. We started with the normal inclusion $S \lhd \Delta(2, 3, 7)$, but S is also normally contained in the obvious group $\Delta(7, 7, 7)$ that has one seventh of the 14-gon as fundamental domain. The corresponding regular $(7, 7, 7)-$dessin lies in the border of the polygon: it has one black vertex, one white vertex, and seven edges. There is even a group $\Delta(3, 3, 7)$ lying between $\Delta(7, 7, 7)$ and $\Delta(2, 3, 7)$ that defines another regular dessin of type (3, 3, 7). The chain of inclusions $S < \Delta(7, 7, 7) < \Delta(3, 3, 7) < \Delta(2, 3, 7)$ means that the corresponding regular dessins are related by refinement.

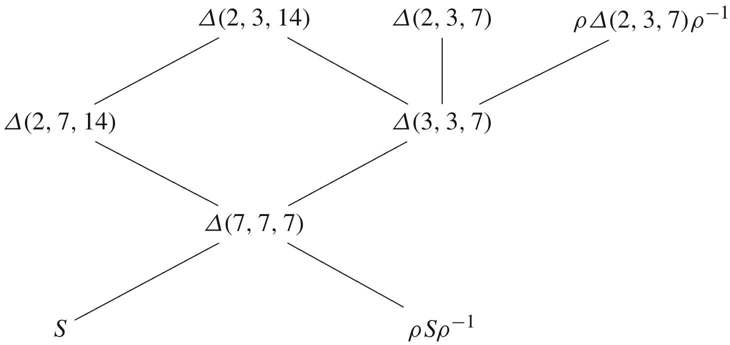
The full diagram of triangle groups lying above S is nevertheless larger. Looking at Singerman's inclusion list we find



(1)

The groups $\Delta(2, 7, 14)$ and $\Delta(2, 3, 14)$ are the index two (therefore normal) extensions of $\Delta(7, 7, 7)$ and $\Delta(3, 3, 7)$ obtained by addition of a new element ρ which is a rotation of angle $2\pi/14$ around the origin. The corresponding dessins of type (2, 7, 14) and (2, 3, 14) are not regular but only uniform (as already noticed in [13]), and are obtained from those of types (7, 7, 7) and (3, 3, 7) by colouring all the vertices with the same colour, say black, and then adding white vertices at the midpoints of the edges.

Conjugation of diagram (1) by ρ fixes all the groups except S and $\Delta(2, 3, 7)$:



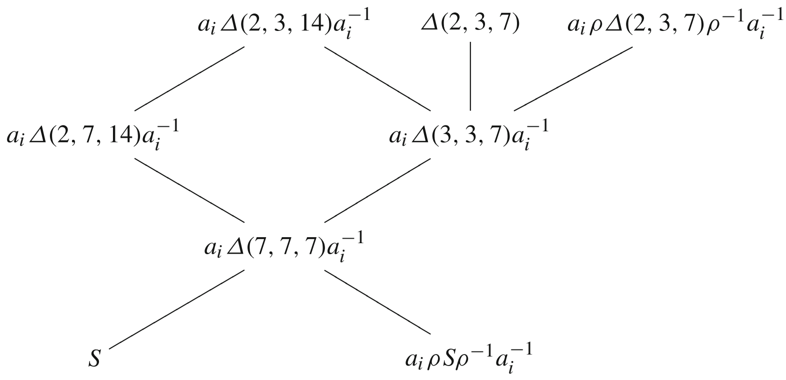
(2)

The inclusion $S < \rho\Delta(2, 3, 7)\rho^{-1}$ corresponds to the uniform dessin $\mathcal{D}'$ described above. Since the normalizer of S is $\Delta(2, 3, 7)$ the inclusion of S in $\rho\Delta(2, 3, 7)\rho^{-1}$ is not normal, hence $\mathcal{D}'$ is not regular.

Now we focus in the group $\Delta(3, 3, 7)$ lying in the middle of diagrams (1) and (2). It is a known fact [5] that a given triangle group of type (3, 3, 7) is contained in precisely two different groups of signature (2, 3, 7) ($\Delta(2, 3, 7)$ and $\rho\Delta(2, 3, 7)\rho^{-1}$ in our case). Reciprocally, any given $\Delta(2, 3, 7)$ contains eight different subgroups of signature (3, 3, 7), all conjugate in $\Delta(2, 3, 7)$.

Let $a_1\Delta(3, 3, 7)a_1^{-1}, \ldots, a_7\Delta(3, 3, 7)a_7^{-1}$ be the seven subgroups of $\Delta(2, 3, 7)$ conjugate to $\Delta(3, 3, 7)$, where $a_i \in \Delta(2, 3, 7)$.

If we conjugate diagram (2) by a_i we get



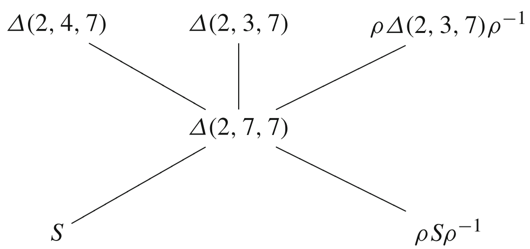
(3)

Note that only $\Delta(2,3,7)$ and S remain fixed by this conjugation, since a_i belongs to $\Delta(2,3,7)$, the normalizer of S.

The inclusion $S < a_i \rho \Delta(2,3,7) \rho^{-1} a_i^{-1}$ induces a new uniform (but not regular) dessin of type $(2,3,7)$ on $S \backslash \mathbf{H}$. It is related to the uniform dessin $\mathcal{D}'$ by the automorphism induced by a_i, and to the regular dessin $\mathcal{D}$ by a hyperbolic rotation of angle $2\pi/14$ around the center of certain face of $\mathcal{D}$.

Macbeath's curve of genus seven. The description of the uniform $(2,3,7)$ dessins on Macbeath curve goes more or less along the same lines as in the case of Klein's quartic. Again the surface group S is included normally in $\Delta(2,3,7)$. The role played by the group $\Delta(3,3,7)$ in Klein's quartic is played here by $\Delta(2,7,7)$. Note that the inclusion $\Delta(2,7,7) < \Delta(2,3,7)$ is also very special (cf. [5]). The number of conjugate subgroups of type $(2,7,7)$ inside $\Delta(2,3,7)$ is nine, and any given $\Delta(2,7,7)$ is contained in two different groups of type $(2,3,7)$. The normalizer of $\Delta(2,7,7)$ is now a $(2,4,7)$–group obtained by adding a rotation ρ of order 4 around any of the points of order 2 in $\Delta(2,7,7)$.

This new element does not normalize $\Delta(2,3,7)$, so conjugation by ρ gives rise to the second group $\rho\Delta(2,3,7)\rho^{-1}$ in which $\Delta(2,7,7)$ is included:



(4)

The inclusion of S inside $\Delta(2,3,7)$ and $\rho\Delta(2,3,7)\rho^{-1}$ determines two non-isomorphic dessins on Macbeath's curve. Once more the second inclusion is not normal, and accordingly the second dessin is uniform but not regular, see Fig. 6.

We can proceed in the same way with the other eight $(2,7,7)$–groups contained inside $\Delta(2,3,7)$ to get diagrams similar to diagram (3). This way we find the nine (isomorphic) uniform dessins predicted by the arithmetic arguments of Sect. 4.

There is obviously as well a uniform dessin of type $(2,4,7)$, as already noticed in [13].

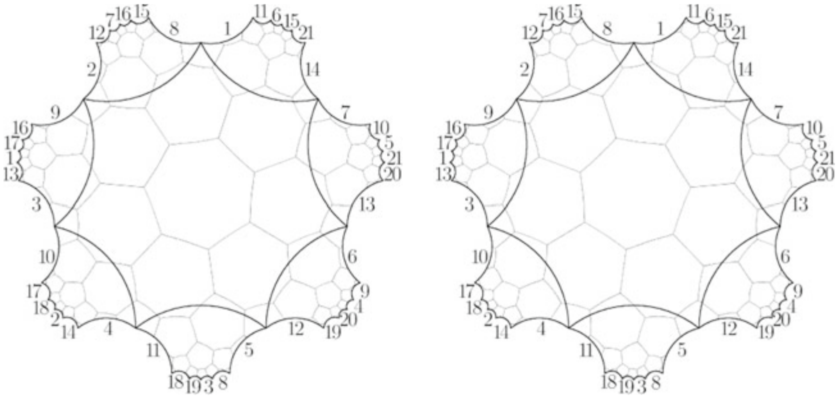


Fig. 6 Face decomposition associated to regular and uniform dessins of type (2, 3, 7) on Macbeath's surface

Macbeath–Hurwitz curves of genus 14. The third example given in Sect. 4 arises from the consideration of the three (torsion free) groups $S_i = \Delta(\pi_i) \lhd \Delta(2,3,7)$ for inequivalent primes π_1, π_2 and π_3 dividing 13 in $\mathbf{Q}(\cos\frac{\pi}{7})$. These groups correspond to three Galois conjugate curves [14] of genus 14 with a regular (2, 3, 7) dessin.

Now for each of these primes, we find $\Delta_0(\pi_i)$ lying between $\Delta(\pi_i)$ and $\Delta(2,3,7)$. Its index inside $\Delta(2,3,7)$ is 14. By Singerman's method for the determination of signatures of subgroups of Fuchsian groups [10] it can be seen that $\Delta_0(\pi_i)$ is a group of signature $\langle 0; 2, 2, 3, 3\rangle$.

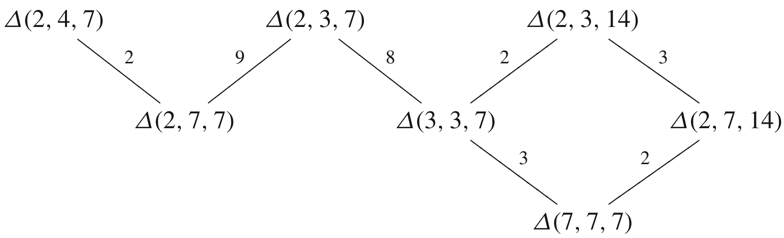
There is again an element ρ_i in the normalizer of $\Delta_0(\pi_i)$ that conjugates $\Delta(2,3,7)$ into a different group. The inclusion of $\Delta(\pi_i)$ inside $\rho_i \Delta(2,3,7)\rho_i^{-1}$ is no longer normal and gives rise to a non-regular uniform dessin on the same Riemann surface.

Moreover, $\Delta(2,3,7)$ contains 14 different subgroups conjugate to $\Delta_0(\pi_i)$. All of them include $\Delta(\pi_i)$, therefore arguing as above we find 14 isomorphic uniform (2, 3, 7) dessins.

6 Commensurability

The previous section has given a first geometric look onto uniform dessins of arithmetic type. Some of the triangle groups involved are not the norm 1 group of a maximal order, but only groups commensurable to it. The number of uniform dessins on a surface $S\backslash\mathbf{H}$, where S is contained in an arithmetic triangle group Δ, depends on the particular relation between Δ and the corresponding norm 1 group. We shall focus on some relevant examples instead of giving complete results for all arithmetic triangle groups.

We begin with the triangle group $\Delta(2,3,7)$ whose commensurable triangle groups can be found in the graph (X) of [16] that we depict here:



(5)

Diagrams (1) and (2) of the last section are contained in (5). As we already know, the group S uniformizing Klein's quartic is contained in $\Delta(7,7,7)$. Moreover, S is a principal congruence subgroup for the prime $\pi \in \mathcal{O}$ dividing the rational prime 7, or equivalently the kernel $\Delta(\pi)$ of the canonical epimorphism of $\Delta(2,3,7)$ onto $\mathrm{PSL}_2(\mathbf{F}_7)$ (since $N(\pi)=7$). Using Singerman's procedure again and considering the (transitive) action of $\Delta(2,3,7)$ on the 8 points of the projective line $\mathbf{P}^1(\mathbf{F}_7)$ it is an easy exercise to see that $\Delta_0(\pi)=\Delta(3,3,7)$, and that its commutator subgroup is precisely $\Delta(7,7,7)$, the preimage of the cyclic subgroup $\left\{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} \mid b \in \mathbf{F}_7\right\}$ of $\mathrm{PSL}_2(\mathbf{F}_7)$.

The remaining groups

$$\Delta(2,3,14)=N(\Delta(3,3,7)) \quad \text{and} \quad \Delta(2,7,4)=N(\Delta(2,7,7))$$

can be arithmetically constructed as the extensions of $\Delta_0(\pi)$ and $\Delta(7,7,7)$ by the Fricke involution.

What do these considerations mean for the existence and the number of different dessins of the same type? First we have to note that for all discrete valuations w of $\mathcal{O}$ not corresponding to the prime $\pi \mid 7$ the triangle groups $\Delta(2,3,7)$, $\Delta(3,3,7)$, $\Delta(7,7,7)$ have the same completion $\mathrm{PSL}_2(\mathcal{O}_w)$, so the situation and in particular the local counting functions d and b do not differ from $\Delta(2,3,7)$ with the single exception of the completion for π itself. Second, recall that $\Delta(2,3,14)$ and $\Delta(7,7,7)$ are uniquely determined as the normalizer and the commutator subgroup of $\Delta(3,3,7)$, and that they have three conjugate copies of $\Delta(2,7,14)$ in between, so it may be sufficient to consider surfaces with different uniform dessins of type $(3,3,7)$. Groups of type $(3,3,7)$ are characterized as norm 1 groups of Eichler orders of level $\mathcal{P}=\langle\pi\rangle$. Therefore the number of uniform $(3,3,7)$-dessins on the surface $S\backslash\mathbf{H}$ is given by the number of edges in the tree $\mathcal{T}(S)$ (see Definition 3), in contrast to the case of $(2,3,7)$-dessins, where one must count vertices. We summarize some possible extensions of Lemmas 4, 7 and Theorem 4 in the following Lemma.

Lemma 8 1. *Let* $\Delta := \Delta(2,3,7)$ *and let* π *be the prime in* $k=\mathbf{Q}(\cos\frac{\pi}{7})$ *dividing* $q=7$. *The surface group* S *is contained in more than one group conjugate to* $\Delta(3,3,7)$ *if and only if* S *is contained in a group conjugate to* $\Delta_0^0(\pi)$.
2. *For* $n>0$ *we have* $d_\pi(\Delta(3,3,7),\Delta(\pi^n))=(q+1)(q^n-1)/(q-1)=\frac{4}{3}(7^n-1)=b_\pi(\Delta(3,3,7),\ 3\cdot 7^{3n-2})$.
3. *If* S *is contained in* d *different groups conjugate to* $\Delta(7,7,7)$, *then it is also contained at least in* d *different groups conjugate to* $\Delta(3,3,7)$. *Equality holds if the minimal congruence subgroup containing* S *is a principal congruence subgroup.*

Remark 4 For $n=1$, $S=\Delta(\pi)$, we have already seen that there are 8 possible dessins of type $(3,3,7)$ on Klein's quartic. They are even regular and all equivalent under automorphisms of the surface, of course. As in Sect. 5, we can always pass from $\Delta(\pi^n)$ to subgroups of small index to obtain examples with many uniform dessins not equivalent under automorphisms.

In the commensurability diagram (5) there is a second branch given by

$$\Delta(2,4,7) \ > \ \Delta(2,7,7) \ < \ \Delta(2,3,7)$$

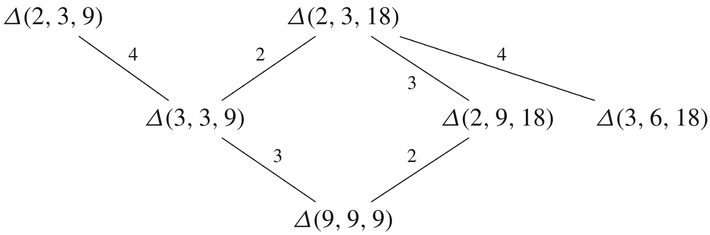
already mentioned in the previous section during the construction of the uniform dessins inside Macbeath's curve. Now let π be the prime 2, inert in the cubic field extension $\mathbf{Q}\left(\cos\frac{\pi}{7}\right)/\mathbf{Q}$, hence with residue class field $\mathbf{F}_8$. Recall from Sect. 4 that $\Delta(2)$ is the surface group of the Macbeath–Hurwitz curve with automorphism group $\mathrm{PSL}_2(\mathbf{F}_8)$ of order

$504 = 9 \cdot 8 \cdot 7$, hence we have a natural action of Δ on the projective line with 9 elements such that $\Delta_0(2)$ will be the subgroup stabilizing one of these points on the projective line. On the other hand, Singerman's method gives the signature of this subgroup defining a transitive action of Δ on nine elements by e.g.

$$\gamma_0 \mapsto (14)(23)(67)(89), \quad \gamma_1 \mapsto (123)(456)(789), \quad \gamma_\infty \mapsto (1697543).$$

The result is $\Delta_0(2) = \Delta(2, 7, 7)$, and the extension by the Fricke involution is just its normalizer $\Delta(2, 4, 7)$. For the counting functions one may draw similar conclusions as in Lemma 8. For example, the surface group S is contained in more than one triangle group of type $(2, 7, 7)$ if and only if it is contained in a group conjugate to $\Delta_0^0(2)$. For another look onto the dessins of type $(2, 7, 7)$ also including noneuclidean cristallographic groups compare [12].

Our second example is given by Takeuchi's diagram (XI). Here the norm 1 group is $\Delta = \Delta(2, 3, 9)$ containing the triangle group $\Delta(3, 3, 9)$ with index 4.



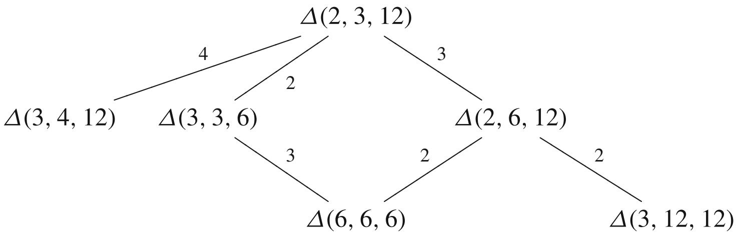
(6)

The quaternion algebra is defined over the cubic field $k := \mathbf{Q}(\cos \frac{\pi}{9})$ and is unramified. In k we have a ramified prime $\pi \mid 3$ of norm 3, and similarly to the inclusion $\Delta(3, 3, 7) < \Delta(2, 3, 7)$ studied above we have here $\Delta_0(\pi) = \Delta(3, 3, 9)$. There has to be an extension of index 2 by the Fricke involution, and in fact $\Delta(3, 3, 9)$ is normalized by the triangle group $\Delta(2, 3, 18)$.[1] The group $\Delta_0(\pi)$ contains $\Delta_0^0(\pi)$ with index $N(\pi) = 3$, and there seems to be a candidate: the subgroup $\Delta(9, 9, 9) < \Delta(3, 3, 9)$. But this group is even not conjugate to $\Delta_0^0(\pi)$: as we have seen in Lemma 3 and its proof, $\Delta_0^0(\pi) = \Delta(\pi)$ and has normalizer $\Delta = \Delta(2, 3, 9)$, whereas $\Delta(9, 9, 9)$ has normalizer $\Delta(2, 3, 18)$. There is no other candidate for a congruence subgroup contained in $\Delta(9, 9, 9)$, so this is a non-congruence subgroup of Δ. Therefore, for all primes $\neq \pi$ dessins of type $(9, 9, 9)$ behave as if $\Delta(9, 9, 9)$ were a norm 1 group, and for the prime π it behaves like those of type $(3, 3, 9)$ because both triangle groups belong to an uniquely determined Eichler order. The same is true for the two other triangle groups $\Delta(3, 6, 18)$ and $\Delta(2, 9, 18)$. They are non-congruence subgroups as well.

Finally, commensurability between triangle groups may come also from the fact that the norm 1 group of a quaternion algebra A is not maximal. This happens if and only if the normalizer of a maximal order $\mathcal{M}$ is a proper extension of the norm 1 group $\mathcal{M}^1$. By [16] it is always a finite extension of 2–power index generated by totally positive units of k and totally positive primes $\pi \in k$ dividing the discriminant $D(A)$. As an example, take the norm 1 group $\Delta = \Delta(3, 3, 6)$ of a quaternion algebra with center $\mathbf{Q}(\sqrt{3})$, extend it by elements in

[1] The first and the last author have to mention that Table 1, case iv, in [5] needs a minor correction: the claim is true that for $\Delta_1 = \Delta(3, n, 3n)$ there is only one supergroup conjugate to $\Delta_2 = \Delta(2, 3, 3n)$ if $n > 3$. However, two such supergroups exist in the case $n = 3$, conjugate under the Fricke involution.

$GL_2(\mathcal{O})$ of determinant $\varepsilon = 2 + \sqrt{3}$, renormalize these elements dividing by $\sqrt{\varepsilon}$ to obtain elements in the extended triangle group $\overline{\Delta} = \Delta(2, 3, 12)$.



(7)

Clearly, every uniform (3, 3, 6)–dessin on a surface with surface group S can be extended to a (2, 3, 12)–dessin, but the converse may fail: a torsion free subgroup $S \subset \overline{\Delta}$ is a subgroup of Δ if and only if it consists of norm 1 elements in $\mathcal{M}$. However, the intersection with Δ has index at most 2 in S, so we may use the fact that Δ is the unique index 2 subgroup of $\overline{\Delta}$ to conclude

Lemma 9 1. *For a surface group $S \subset \Delta$ we have $d(\Delta, S) = d(\overline{\Delta}, S)$.*
2. *For a surface group $S \subset \overline{\Delta}$ we have $d(\Delta, S \cap \Delta) = d(\overline{\Delta}, S)$ where $(S : S \cap \Delta) \leq 2$.*

As an exercise, the reader may prove that the triangle group $\Delta(3, 4, 12)$, an index 4 subgroup of $\overline{\Delta}$, is an index 2 extension of the congruence subgroup $\Delta_0(\sqrt{3})$. The group $\Delta(6, 6, 6)$ is the unique index 3 normal subgroup of the norm 1 group $\Delta = \Delta(3, 3, 6)$, therefore it must be the principal congruence subgroup $\Delta(1 + \sqrt{3})$ since the prime $1 + \sqrt{3}$ is the unique discriminant divisor of the associated quaternion algebra [16]. It is a prime of norm 2, therefore the residue class field of the (unique!) maximal order is $\mathbf{F}_4$, and $\Delta/\Delta(1 + \sqrt{3})$ has to be isomorphic to its multiplicative group. An obvious variant of Theorem 4 for this triangle group is therefore

Lemma 10 *A Fuchsian group S is contained in more than one conjugate of the triangle group $\Delta(6, 6, 6)$ if and only if it is contained in a congruence subgroup $\Delta(1+\sqrt{3}) \cap \Delta_0(\pi)$ of the group $\Delta = \Delta(3, 3, 6)$ for a prime π of the field $\mathbf{Q}(\sqrt{3})$ not equivalent to $1 + \sqrt{3}$.*

The triangle group $\Delta(2, 6, 12)$ can be obtained extending $\Delta(6, 6, 6)$ by elements of norm ε. It is a congruence subgroup of $\Delta(2, 3, 12)$, whereas $\Delta(3, 12, 12)$ seems to be a non-congruence subgroup.

7 A genus 2 surface with two uniform dessins

A complete list of all (isomorphism classes of) uniform dessins in genus 2 is given in [13]. It is however not obvious if and when two such dessins—even if they are of the same type—may belong to the same surface. The arithmetic considerations of Sect. 6 about the $\Delta(2, 3, 9)$ group will allow to construct an example of a genus 2 surface with two non-isomorphic uniform dessins of the same type.

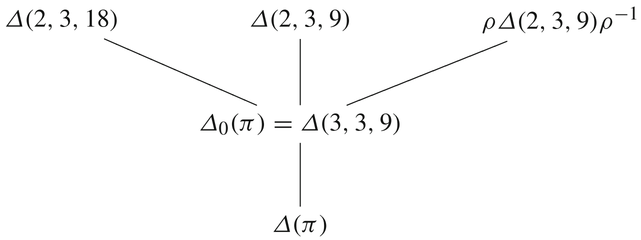
Let us consider the ramified prime $\pi \in \mathbf{Q}(\cos \frac{\pi}{9})$ dividing 3.

For $\Delta = \Delta(2, 3, 9)$ we have the following chain of inclusions:

$$\Delta \overset{4}{>} \Delta_0(\pi) \overset{3}{>} \Delta(\pi),$$

where $\Delta_0(\pi) = \Delta(3, 3, 9)$ and $\Delta(\pi)$ is the principal congruence subgroup of level π, a Fuchsian group of signature $\langle 0\,;\; 3, 3, 3, 3\rangle$. Moreover, since $q = 3$ we have $\Delta(\pi) \simeq \Delta_0(\pi^2)$ (see Remark 2).

By the arithmetic theory developed before, the Fricke involution conjugates $\Delta(2, 3, 9)$ into another group $\rho\Delta(2, 3, 9)\rho^{-1}$ such that:



(8)

Once again ρ is an extra rotation—of order 2 around a fixed point of order 9—inside $\Delta(2, 3, 18)$, the normalizer of $\Delta(3, 3, 9)$. Let us note that conjugation by the Fricke involution gives precisely the isomorphism between $\Delta(\pi)$ and $\Delta_0(\pi^2) = \rho\Delta(\pi)\rho^{-1}$.

Now by Theorem 4 every surface group inside $\Delta(3, 3, 9)$ will have at least two (2, 3, 9) dessins. By the list in [13] we know that in genus 2 there are 4 different dessins of this type. For two of them it can be seen, by computing the monodromies and constructing a fundamental domain, that the Fricke involution is an automorphism of the surface, and so the two dessins arising from the arithmetic construction are isomorphic (see also [4]).

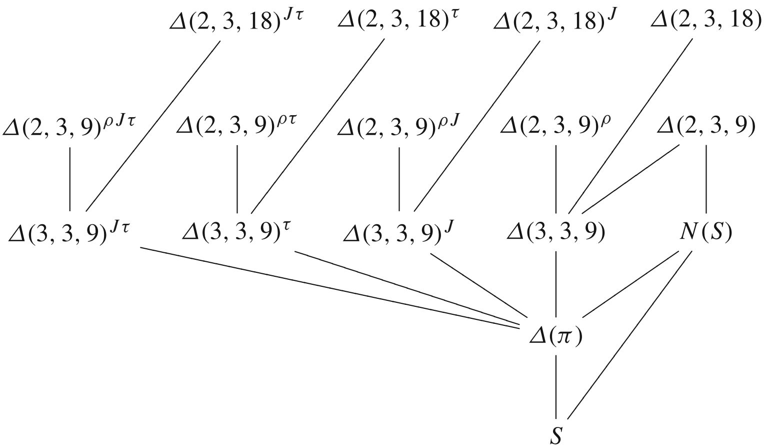
The other two are the dual dessins considered in [13], Section 11(d). To find its surface group we can follow once more Singerman's procedure, and it can be seen that it is possible to find a (normal) torsion free subgroup S of index 3 in $\Delta(\pi)$. The indices $(\Delta(3, 3, 9) : S) = 9$ and $(\Delta(2, 3, 9) : S) = 36$ tell us that it corresponds indeed to a genus 2 surface.

The monodromies of the two dessins induced by $\Delta(2, 3, 9)$ and $\rho\Delta(2, 3, 9)\rho^{-1}$ are non-conjugate inside $\mathcal{S}_{36}$ so they are not isomorphic as we already knew, neither are they even equivalent under automorphisms or renormalization. (David Singerman kindly informed the authors that this uniform non-regular dessin and the uniform non-regular dessin on Klein's quartic given in Fig. 5 were already found by R.I. Syddall in his unpublished PhD thesis [15].)

By [4] we know that the automorphism group of this surface $\mathrm{Aut}(S\backslash\mathbf{H}) \simeq N(S)/S$ is generated by the hyperelliptic involution J and two automorphisms τ and σ_3 of order 2 and 3 respectively. From the arithmetic point of view, we can even say that $\langle S, \widetilde{\sigma}_3\rangle \simeq \Delta(\pi)$, where $\widetilde{\sigma}_3$ denotes the lift of σ_3 to $\mathbf{H}$.

The lifts of all these automorphisms lie inside $\Delta(2, 3, 9)$, but $\widetilde{J}, \widetilde{\tau}, \widetilde{J\tau} \in N(S)$ do not belong to $\rho\Delta(2, 3, 9)\rho^{-1}$. Conjugation of $\rho\Delta(2, 3, 9)\rho^{-1}$ by each of these elements will determine another (2, 3, 9)-dessin isomorphic to the second one.
The same can be applied to $\Delta(3, 3, 9)$ and $\Delta(2, 3, 18)$. In particular $\Delta(3, 3, 9)$, $\widetilde{J}(\Delta(3, 3, 9))\widetilde{J}^{-1}$, $\widetilde{\tau}(\Delta(3, 3, 9))\widetilde{\tau}$ and $\widetilde{J\tau}(\Delta(3, 3, 9))(\widetilde{J\tau})^{-1}$ are the four different (3, 3, 9) groups lying below a given $\Delta(2, 3, 9)$ ([5], p. 9, Thm. 6).

The following diagram of inclusions shows all the dessins (modulo renormalization) in this surface. The notation G^σ stands for conjugation by σ:



(9)

By localization, the different (2, 3, 9) and (3, 3, 9) groups can be seen as generating the local maximal orders and Eichler orders of level $\mathcal{P}$ respectively, containing $S_\pi \lhd (\Delta(\pi))_\pi$ (see the vertices and edges of the subtree in Figs. 7, 8).

To sum up, there are (up to renormalization) four different (3, 3, 9) dessins on the surface $S\backslash\mathbf{H}$ studied here, forming one orbit under the automorphism group $N(S)/S \simeq D_3 \times C_2$ act-

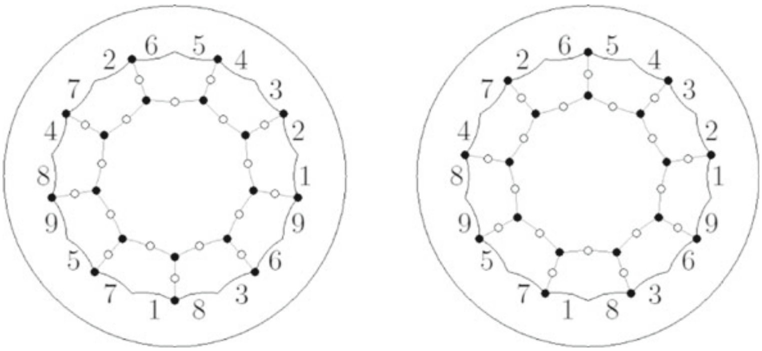


Fig. 7 Two non-isomorphic uniform dessins of type (2, 3, 9) in the same surface

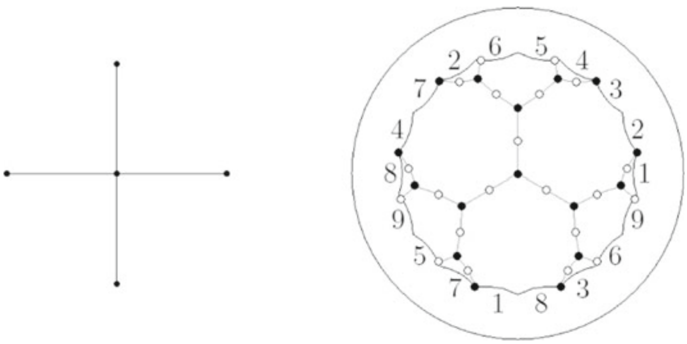


Fig. 8 Subtree $\mathcal{T}(\Delta_\pi(\pi))$ and the image of the second dessin under the hyperelliptic involution J

ing on the edges of the subtree given in Fig. 8. On the other hand one has four (2, 3, 9) dessins equivalent under the automorphism group plus one stabilized by $N(S)/S$, corresponding to the mid-vertex of the subtree, not isomorphic to the others.

Remark 5 According to [13] an equation for $S\backslash\mathbf{H}$ is $y^3 = (x-1)(x^3-1)$. We have found that $y^2 = x^6 + 8x^3 + 4$ is a hyperelliptic model of this surface.

Acknoweldgments Ernesto Girondo was partially supported by grants MTM2006-28257-E and CCG08-UAM/ESP-4145. David Torres-Teigell was partially supported by an FPU grant of the Ministerio de Ciencia e Innovación. Jürgen Wolfart was supported by Deutsche Forschungsgemeinschaft grants Wo 199/3 and Wo 199/4.

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